First name:____________________

Last name: KEY

Student ID:____________________

Signature:____________________

Read before you start:

- There are five questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 100 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS unless otherwise stated.

Q1	Q2	Q3	Q4	Q5	Total

Q1.

Let $Q \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix with distinct eigenvalues. Consider the following nonlinear system.

$$\dot{x} = [xx^T - Q]x \,.$$

a) Find all the equilibria of this system.

b) Determine the (local) stability of each of these equilibria through linearization.

Note that $\frac{\partial}{\partial x}\{xx^T x\} = \|x\|^2 I + 2xx^T$.

Let $\lambda_1, \lambda_2, \dots, \lambda_n > 0$ be the eigenvalues and $v_1, v_2, \dots, v_n$ be the corresponding eigenvectors of Q.

a) $0 = [xx^T - Q]x = xx^Tx - Qx = \|x\|^2 x - Qx$

$\Rightarrow Qx = \|x\|^2 x \quad (1)$

(1) implies that either $x = 0$ or x is an eigenvector $x = \alpha v_i$. To find α use (1).

$\|\alpha v_i\|^2 (\alpha v_i) = Q(\alpha v_i) = \alpha \lambda_i v_i$

$\Rightarrow \|\alpha v_i\|^2 = \lambda_i \Rightarrow \alpha = \mp \frac{\sqrt{\lambda_i}}{\|v_i\|}$

Hence, the system has $2n+1$ equilibria:

$$\boxed{x_e \in \left\{0, \mp \frac{\sqrt{\lambda_1}}{\|v_1\|} v_1, \dots, \mp \frac{\sqrt{\lambda_n}}{\|v_n\|} v_n\right\}}$$

b) $\frac{\partial}{\partial x}\{xx^Tx - Qx\} = \|x\|^2 I + 2xx^T - Q$

$\underline{x = 0}$: $A = \frac{\partial}{\partial x}\{xx^Tx - Qx\}\Big|_{x=0} = -Q$

all the eigenvalues of $-Q$ are negative because Q is pos. def.

$$\boxed{\Rightarrow x = 0 \text{ is (locally) asy. stable}}$$

$\underline{x = \mp \frac{\sqrt{\lambda_i}}{\|v_i\|} v_i}$: $A = \lambda_i I + \frac{2\lambda_i}{\|v_i\|^2} v_i v_i^T - Q$

Note that $A v_i = \lambda_i v_i + 2\lambda_i v_i - Q v_i = 2\lambda_i v_i$

Hence A has a positive eigenvalue $\lambda = 2\lambda_i$.

$$\boxed{\Rightarrow x = \mp \frac{\sqrt{\lambda_i}}{\|v_i\|} v_i \text{ are unstable}}$$

Q2.

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$, and $K \in \mathbb{R}^{k\times n}$. Also, let $P \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Prove the following statements.

a) If the system $\dot{x} = [A - BK]x$ is exponentially stable, then the system $\dot{x} = Ax + Bu$ is stabilizable.

b) If the pair (A, B) is controllable and the equation $APA^T - P + BB^T = 0$ holds, then the system $x^+ = Ax$ is exponentially stable.

a) Suppose not. That is, $[A-BK]$ is exp. stable yet (A,B) is not stabilizable. Then we can find an eigenvector v such that

$A^Tv = \lambda v$ & $Re\{\lambda\} \geq 0$ & $B^Tv = 0$

Now, $[A-BK]^Tv = (A^T - K^TB^T)v$

$$= A^Tv - K^TB^Tv \quad B^Tv = 0$$
$$= A^Tv$$
$$= \lambda v$$

$\Rightarrow$ λ is an eigenvalue of $A-BK$

Since $Re\{\lambda\} \geq 0$, $[A-BK]$ cannot be exp. stable. Result follows by contradiction. ▨

b) Let λ be an arbitrary eigenvalue of A. Let v be the eigenvector of A^T for λ. That is, $A^Tv = \lambda v$.

Now, $0 = v^*(APA^T - P + BB^T)v$

$$= (A^Tv)^* P A^Tv - v^*Pv + v^*BB^Tv$$
$$= |\lambda|^2 v^*Pv - v^*Pv + \|B^Tv\|^2$$
$$= (|\lambda|^2 - 1)\underbrace{v^*Pv}_{>0 \text{ because } P>0} + \underbrace{\|B^Tv\|^2}_{>0 \text{ because } (A,B) \text{ cont.}}$$

$\Rightarrow |\lambda|^2 - 1 < 0 \Rightarrow |\lambda| < 1 \Rightarrow$ exp. stab. ▨

Q3.

Consider the following system

$$\dot{x} = \left(\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} K\right) x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$
$$y = [4 \ \ 3]x$$

where $K = [k_1 \ \ k_2]$ represents the state feedback gain.

a) Obtain the controllable decomposition for the open-loop ($K = 0$) system.

b) If possible, find a gain K such that the closed-loop system is internally stable.

c) If possible, find a gain K such that the closed-loop system is BIBO stable and compute the resulting closed-loop transfer function.

a) $\text{range}\,[B \ \ AB] = \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

$\Rightarrow$ let $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ & $\upsilon = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T = [v \ \ \upsilon] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ is the coordinate change matrix.

$z = T^{-1}x \Rightarrow \dot{z} = T^{-1}ATz + T^{-1}Bu$

$y = CTz$

$$T^{-1}AT = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$$

$$T^{-1}B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$CT = [4 \ \ 3]\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = [1 \ \ 3]$$

$$\Rightarrow \boxed{\begin{aligned} \dot{z} &= \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= [1 \ \ 3] z \end{aligned}}$$

b) We can write for $K \neq 0$

$$\dot{z} = \left(\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\bar{K}\right) z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [1 \ \ 3] z$$

where $\bar{K} = KT \Rightarrow K = \bar{K}T^{-1}$

Choose, for instance, $\bar{K} = [4 \ \ 0]$

$$\Rightarrow \bar{A} - \bar{B}\bar{K} = \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}[4 \ \ 0] = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$$

$\lambda_1 = -1$ & $\lambda_2 = 0 \Rightarrow$ internally stable

$$\Rightarrow K = \bar{K}T^{-1} = [4 \ \ 0]\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \boxed{[4 \ \ 0]}$$

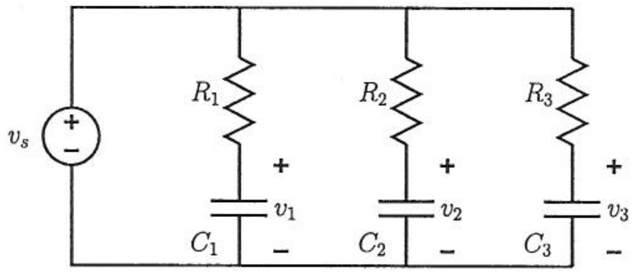
c) We can use the same $K = [4 \ \ 0]$, then

$$\left.\begin{aligned} \dot{z} &= \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= [1 \ \ 3] z \end{aligned}\right\} \Rightarrow \boxed{\frac{Y(s)}{U(s)} = \frac{1}{s+1}}$$

Since the only pole of TF is at $\lambda = -1 < 0$, the closed-loop system is BIBO stable.

Q4.

Consider the following third-order circuit where the capacitor voltages v_i are related to the input voltage v_s through $R_i C_i \dot{v}_i + v_i = v_s$ for $i = 1, 2, 3$.



$$R_1 = 2\,\text{k}\Omega, \quad C_1 = \frac{1}{4}\,\text{mF}$$

$$R_2 = 2\,\text{k}\Omega, \quad C_2 = \frac{1}{6}\,\text{mF}$$

a) Take $C_3 = \frac{1}{12}\,\text{mF}$. Suppose the initial capacitor voltages are $v_1(0) = 1\,\text{V}$, $v_2(0) = -2\,\text{V}$, and $v_3(0) = 3\,\text{V}$. We want to completely discharge all three capacitors within $T = 137\text{msec}$, i.e., the goal is to achieve $v_i(T) = 0$ for $i = 1, 2, 3$. For what value(s) of the resistance $R_3 > 0$ is our goal impossible?

b) Take $C_3 = \frac{1}{24}\,\text{mF}$ and $R_3 = 8\,\text{k}\Omega$. Suppose that the capacitors are initially uncharged, i.e., $v_i(0) = 0$ for $i = 1, 2, 3$. Determine whether the below given triplets of capacitor voltages are attainable. ($T = 137\text{msec.}$)

(i) $(v_1(T), v_2(T), v_3(T)) = (4, -3, 5)$.
(ii) $(v_1(T), v_2(T), v_3(T)) = (-6, -6, 5)$.
(iii) $(v_1(T), v_2(T), v_3(T)) = (0, 7, 7)$.
(iv) $(v_1(T), v_2(T), v_3(T)) = (3, 9, 3)$.
(v) $(v_1(T), v_2(T), v_3(T)) = (12, 12, 12)$.

a)

$$\dot{v} = \begin{bmatrix} -(R_1C_1)^{-1} & 0 & 0 \\ 0 & -(R_2C_2)^{-1} & 0 \\ 0 & 0 & -(R_3C_3)^{-1} \end{bmatrix} v + \begin{bmatrix} (R_1C_1)^{-1} \\ (R_2C_2)^{-1} \\ (R_3C_3)^{-1} \end{bmatrix} v_s$$

let $w_3 = (R_3C_3)^{-1} \neq 0$, Then

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -w_3 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 2 \\ 3 \\ w_3 \end{bmatrix}$$

controllability matrix ?

$$C = [B \;\; AB \;\; A^2B] = \begin{bmatrix} 2 & -4 & 8 \\ 3 & -9 & 27 \\ w_3 & -w_3^2 & w_3^3 \end{bmatrix}$$

uncontrollability $\Rightarrow$ rank $C < 3$

$\Rightarrow w_3 = 2$ or $w_3 = 3$

($w_3 = 0$ impossible!)

$\underline{\underline{w_3 = 2}}$ $\quad$ range $C = \text{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} \not\ni \begin{bmatrix}1\\-2\\3\end{bmatrix}$

$\underline{\underline{w_3 = 3}}$ $\quad$ range $C = \text{span}\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\} \not\ni \begin{bmatrix}1\\-2\\3\end{bmatrix}$

$w_3 = 2$ or $3 \Rightarrow \boxed{R_3 = 6\,\text{k}\Omega \text{ or } 4\,\text{k}\Omega}$

b) $w_3 = \dfrac{1}{R_3C_3} = 3$

$\Rightarrow$ range $C = \text{span}\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

$\Rightarrow$ $\boxed{\text{only (iii) \& (v) are possible}}$

Q5.

Determine whether each of the following statements is true (T) or false (F). (No explanation is required.)

a) If (A, B) is a controllable pair, so is (A^T, B).

b) If (A, B) is a controllable pair, so is (A, BB^T).

c) If (A, B) is a controllable pair, so is $(A - BK, B)$.

d) If a continuous-time LTV system is reachable on some interval $[t_0, t_1]$, it is controllable on the same interval.

e) Any system $\dot{x} = Ax + Bu$, $y = Cx$ can be made BIBO stable by state feedback. I.e., we can always find an appropriate feedback gain K such that the closed-loop system $\dot{x} = (A - BK)x + Bu$, $y = Cx$ is BIBO stable.

f) A second-order system $\dot{x} = Ax + Bu$, $y = Cx$ with impulse response $h(t) = \cos(t)$ must be controllable.

g) If the linearization of a nonlinear system $\dot{x} = f(x)$ at an equilibrium point x_e is unstable, then x_e cannot be asymptotically stable for $\dot{x} = f(x)$.

h) If the linearization of a nonlinear system $\dot{x} = f(x)$ at an equilibrium point x_e is asymptotically stable, then x_e cannot be unstable for $\dot{x} = f(x)$.

Your answer:

a	b	c	d	e	f	g	h
F	T	T	T	T	T	F	T