First name:______________________________

Last name:______________________________

Student ID:______________________________

Signature:______________________________

Read before you start:

- There are five questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 100 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS unless otherwise stated.

Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Total**

Q1.

Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix with distinct eigenvalues. Consider the following nonlinear system.

$$\dot{x} = [xx^T - Q]x\,.$$

a) Find all the equilibria of this system.

b) Determine the (local) stability of each of these equilibria through linearization.

Note that $\dfrac{\partial}{\partial x}\{xx^T x\} = \|x\|^2 I + 2xx^T$.

Q2.

Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$, and $K \in \mathbb{R}^{k \times n}$. Also, let $P \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix. Prove the following statements.

a) If the system $\dot{x} = [A - BK]x$ is exponentially stable, then the system $\dot{x} = Ax + Bu$ is stabilizable.

b) If the pair (A, B) is controllable and the equation $APA^T - P + BB^T = 0$ holds, then the system $x^+ = Ax$ is exponentially stable.

Q3.

Consider the following system

$$\begin{aligned} \dot{x} &= \left(\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} K \right) x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u \\ y &= \begin{bmatrix} 4 & 3 \end{bmatrix} x \end{aligned}$$

where $K = [\, k_1 \;\; k_2 \,]$ represents the state feedback gain.

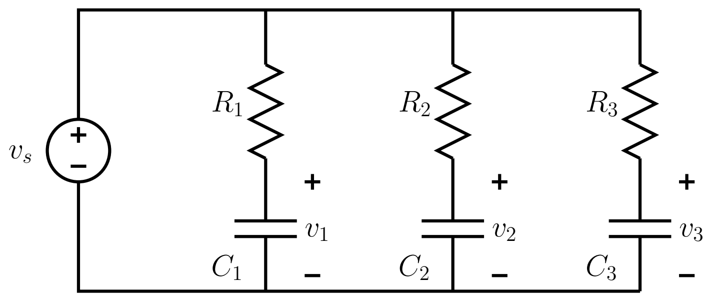
a) Obtain the controllable decomposition for the open-loop ($K = 0$) system.

b) If possible, find a gain K such that the closed-loop system is internally stable.

c) If possible, find a gain K such that the closed-loop system is BIBO stable and compute the resulting closed-loop transfer function.

Q4.

Consider the following third-order circuit where the capacitor voltages v_i are related to the input voltage v_s through $R_i C_i \dot{v}_i + v_i = v_s$ for $i = 1, 2, 3$.



$$R_1 = 2\,\text{k}\Omega\,,\ \ C_1 = \frac{1}{4}\,\text{mF}$$

$$R_2 = 2\,\text{k}\Omega\,,\ \ C_2 = \frac{1}{6}\,\text{mF}$$

a) Take $C_3 = \frac{1}{12}\,\text{mF}$. Suppose the initial capacitor voltages are $v_1(0) = 1\,\text{V}$, $v_2(0) = -2\,\text{V}$, and $v_3(0) = 3\,\text{V}$. We want to completely discharge all three capacitors within $T = 137$msec, i.e., the goal is to achieve $v_i(T) = 0$ for $i = 1, 2, 3$. For what value(s) of the resistance $R_3 > 0$ is our goal impossible?

b) Take $C_3 = \frac{1}{24}\,\text{mF}$ and $R_3 = 8\,\text{k}\Omega$. Suppose that the capacitors are initially uncharged, i.e., $v_i(0) = 0$ for $i = 1, 2, 3$. Determine whether the below given triplets of capacitor voltages are attainable. ($T = 137$msec.)

(i) $(v_1(T),\, v_2(T),\, v_3(T)) = (4,\, -3,\, 5)$.

(ii) $(v_1(T),\, v_2(T),\, v_3(T)) = (-6,\, -6,\, 5)$.

(iii) $(v_1(T),\, v_2(T),\, v_3(T)) = (0,\, 7,\, 7)$.

(iv) $(v_1(T),\, v_2(T),\, v_3(T)) = (3,\, 9,\, 3)$.

(v) $(v_1(T),\, v_2(T),\, v_3(T)) = (12,\, 12,\, 12)$.

Q5.

Determine whether each of the following statements is true (T) or false (F). (No explanation is required.)

a) If (A, B) is a controllable pair, so is (A^T, B).

b) If (A, B) is a controllable pair, so is (A, BB^T).

c) If (A, B) is a controllable pair, so is $(A - BK, B)$.

d) If a continuous-time LTV system is reachable on some interval $[t_0, t_1]$, it is controllable on the same interval.

e) Any system $\dot{x} = Ax + Bu$, $y = Cx$ can be made BIBO stable by state feedback. I.e., we can always find an appropriate feedback gain K such that the closed-loop system $\dot{x} = (A - BK)x + Bu$, $y = Cx$ is BIBO stable.

f) A second-order system $\dot{x} = Ax + Bu$, $y = Cx$ with impulse response $h(t) = \cos(t)$ must be controllable.

g) If the linearization of a nonlinear system $\dot{x} = f(x)$ at an equilibrium point x_e is unstable, then x_e cannot be asymptotically stable for $\dot{x} = f(x)$.

h) If the linearization of a nonlinear system $\dot{x} = f(x)$ at an equilibrium point x_e is asymptotically stable, then x_e cannot be unstable for $\dot{x} = f(x)$.

Your answer:

a	b	c	d	e	f	g	h