First name:______________________________

Last name: KEY

Student ID:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is open-book.
- No calculator is allowed.
- The duration of the examination is 100 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS.

Q1	Q2	Q3	Q4	Total

Q1.

Consider the below system.

$$\dot{x} = \left[\begin{array}{cc|c} \overset{A_c}{\boxed{\begin{matrix}-4 & 3\\ -6 & 5\end{matrix}}} & & \begin{matrix}1\\0\end{matrix} \\ \begin{matrix}0 & 0\end{matrix} & & \underset{A_u}{\boxed{-3}}\end{array}\right] x + \overset{B_c}{\begin{bmatrix} \boxed{\begin{matrix}0\\1\end{matrix}} \\ 0\end{bmatrix}} u \quad = Ax + Bu$$

$$y = \left[\overset{C_c}{\boxed{-2 \quad 1}} \quad 7\right] x.$$

(a) Is this system stable in the sense of Lyapunov?

(b) Is this system controllable?

(c) Is this system stabilizable?

(d) Is this system BIBO stable?

a) $|sI - A_c| = \begin{vmatrix} s+4 & -3 \\ 6 & s-5 \end{vmatrix} = (s+1)(s-2)$

$\lambda(A) = \lambda(A_c) \cup \lambda(A_u)$ since A block triangular

$= \{-1, 2, -3\}$

Due to positive eigenvalue at $\lambda = 2$, system is $\boxed{\text{unstable}}$

b) Let $x = [x_1 \ x_2 \ x_3]^T$. Then $\dot{x}_3 = -3x_3$.

There is no way we can control x_3.

$\Rightarrow$ $\boxed{\text{uncontrollable}}$

c) $\operatorname{rank}[B_c \ A_cB_c] = \operatorname{rank}\begin{bmatrix} 0 & 3 \\ 1 & 5 \end{bmatrix} = 2$

$\Rightarrow$ (A_c, B_c) controllable

$\Rightarrow$ system is in cont. decomp. form

Since $\lambda(A_u) = -3 < 0$, $\boxed{\text{stabilizable}}$

d) $TF = C_c(sI - A_c)^{-1}B_c$

$$= \frac{[-2 \ \ 1]\begin{bmatrix} s-5 & 3 \\ -6 & s+4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{(s+1)(s-2)} = \frac{\cancel{s-2}}{(s+1)\cancel{(s-2)}} = \frac{1}{s+1}$$

$\operatorname{Re}\{\lambda_i(TF)\} = -1 < 0 \Rightarrow$ $\boxed{\text{BIBO stable}}$

Q2.

Consider the following third order system with scalar input and scalar output

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= [\alpha \;\; \beta \;\; 2]x \end{aligned}$$

whose full-state impulse response, i.e., $h(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}B\}$ is given below.

$$h(t) = \begin{bmatrix} e^t + e^{-t} \\ 2 + e^{-t} \\ 1 - e^t \end{bmatrix}$$

(a) Find B.

(b) Compute the controllability matrix.

(c) Does the vector $\bar{x} = [1 \;\; 0 \;\; 1]^T$ belong to the reachable subspace?

(d) Find α and β so that the system is BIBO stable.

$h(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}B\} = e^{At}B$

$\Rightarrow \dot{h}(t) = e^{At}AB \quad \& \quad \ddot{h}(t) = e^{At}A^2B$

a) $B = h(0) = \boxed{[2 \;\; 3 \;\; 0]^T}$

b) $[B \;\; AB \;\; A^2B] = [h(0) \;\; \dot{h}(0) \;\; \ddot{h}(0)]$

$$= \boxed{\begin{bmatrix} 2 & 0 & 2 \\ 3 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}}$$

c) $\det [B \;\; AB \;\; A^2B] = -2 \neq 0 \Rightarrow$ system controllable

$\Rightarrow \boxed{\text{YES}}$

d) (output) impulse response $g(t) = \mathcal{L}^{-1}\{C(sI-A)^{-1}B\}$

$\Rightarrow g(t) = [\alpha \;\; \beta \;\; 2]\, h(t)$

$= \alpha(e^t + e^{-t}) + \beta(2 + e^{-t}) + 2 - 2e^t$

$= (\alpha - 2)e^t + (2\beta + 2)\cdot 1 + (\alpha + \beta)e^{-t}$

BIBO stability $\Rightarrow \int_0^\infty |g(t)|dt < \infty$

$\Rightarrow \boxed{\alpha = 2} \quad \& \quad \boxed{\beta = -1}$

Q3.

Consider the below system which is known to be uncontrollable but stabilizable.

$$\dot{x} = \begin{bmatrix} -5 & 4 \\ -6 & 5 \end{bmatrix} x + \begin{bmatrix} 2 \\ \alpha \end{bmatrix} u$$

(a) Find α.

(b) Obtain the controllable decomposition $\dot{z} = \bar{A}z + \bar{B}u$ via the transformation $z = T^{-1}x$, where the columns of T are the eigenvectors of A.

(c) Choose a diagonal $\bar{P} > 0$ satisfying the Lyapunov inequality $\bar{A}\bar{P} + \bar{P}\bar{A}^T - \bar{B}\bar{B}^T < 0$.

(d) Choose a gain $K \in \mathbb{R}^{1\times 2}$ in terms of $\bar{B}$, $\bar{P}$, T such that the feedback law $u = -Kx$ makes the closed-loop system exponentially stable. *Do not compute K here; just give its expression.*

a) $|sI - A^T| = \begin{vmatrix} s+5 & 6 \\ -4 & s-5 \end{vmatrix} = s^2 - 1 = (s-1)(s+1)$

$\Rightarrow \lambda_1 = 1$ (unstable) & $\lambda_2 = -1$ (stable)

Eigenvectors: $A^T w_1 = \lambda_1 w_1$ & $A^T w_2 = \lambda_2 w_2$

uncontrollable $\Rightarrow B^T w_1 = 0$ or $B^T w_2 = 0$ (1)

stabilizable $\Rightarrow B^T w_1 \neq 0$ (2)

(1) & (2) $\Rightarrow B^T w_2 = 0$

$0 = (\lambda_2 I - A^T) w_2 = \begin{bmatrix} 4 & 6 \\ -4 & -6 \end{bmatrix} w_2 \Rightarrow w_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$0 = B^T w_2 = \begin{bmatrix} 2 & \alpha \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = 6 - 2\alpha \Rightarrow \boxed{\alpha = 3}$

b) Let $Av_1 = \lambda_1 v_1$ & $Av_2 = \lambda_2 v_2$

$0 = [\lambda_1 I - A] v_1 = \begin{bmatrix} 6 & -4 \\ 6 & -4 \end{bmatrix} v_1 \Rightarrow v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$0 = [\lambda_2 I - A] v_2 = \begin{bmatrix} 4 & -4 \\ 6 & -6 \end{bmatrix} v_2 \Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

cont subspace $=$ range $[B \;\; AB] =$ range $\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$

$= \text{span}\{v_1\}$

$\Rightarrow T = [v_1 \;\; v_2] = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$

$\Rightarrow \bar{A} = T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

& $\bar{B} = T^{-1}B = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\Rightarrow \boxed{\dot{z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u}$

c) $\bar{P} = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$ with $p_1 > 0$ & $p_2 > 0$

$\Rightarrow 0 > \bar{A}\bar{P} + \bar{P}\bar{A}^T - \bar{B}\bar{B}^T = \begin{bmatrix} 2p_1 - 1 & 0 \\ 0 & -2p_2 \end{bmatrix}$

$\Rightarrow 2p_1 - 1 < 0$ & $-2p_2 < 0$

we can choose $\boxed{\bar{P} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}}$

d) $0 > \bar{A}\bar{P} + \bar{P}\bar{A}^T - \bar{B}\bar{B}^T$ & $\bar{K} := \frac{1}{2}\bar{B}^T\bar{P}^{-1}$

$= (\bar{A} - \bar{B}\bar{K})\bar{P} + \bar{P}(\bar{A} - \bar{B}\bar{K})^T$

$u = -\bar{K}z = -\bar{K}T^{-1}x =: -Kx$

$\Rightarrow \boxed{K = \frac{1}{2}\bar{B}^T\bar{P}^{-1}T^{-1}}$

Q4.

Consider the predator-prey equations

$$\left.\begin{aligned}\dot{x}_1 &= x_1(1-x_2)\\ \dot{x}_2 &= -x_2(1-x_1).\end{aligned}\right\} =: f(x)$$

(a) Find all the equilibrium points of this system.

(b) At each equilibrium obtain the linearization.

(c) What can (based on the linearization) be said about the local stability of each equilibrium?

a) $f(x) = 0 \Rightarrow x_1(1-x_2) = 0$ & $-x_2(1-x_1) = 0$

$\Rightarrow$ two equilibria $\boxed{x_{eq} \in \{[0\ 0]^T, [1\ 1]^T\}}$

b) $$\frac{\partial f}{\partial x} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} 1-x_2 & -x_1 \\ x_2 & x_1-1 \end{bmatrix}$$

$x_{eq} = [0\ 0]^T$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \boxed{\dot{\xi} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \xi}$$

$x_{eq} = [1\ 1]^T$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \boxed{\dot{\xi} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \xi}$$

c) $x_{eq} = [0\ 0]^T$

$$\lambda\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\} \Rightarrow \lambda_1 = 1 \ \& \ \lambda_2 = -1$$

$\lambda_1 > 0 \Rightarrow$ $\boxed{\text{unstable}}$

$x_{eq} = [1\ 1]^T$

$$\lambda\left\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\} \Rightarrow \lambda_1 = j \ \& \ \lambda_2 = -j$$

$Re\{\lambda_i\} = 0$ for $i = 1, 2$ $\Rightarrow$ $\boxed{\text{inconclusive}}$