First name:____________________

Last name: KEY

Student ID:____________________

Signature:____________________

Read before you start:

- There are four questions.
- The examination is open-book.
- No calculator is allowed.
- The duration of the examination is 100 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS.

Q1	Q2	Q3	Q4	Total
20	20	30	30	

Q1.

Let $P \in \mathbb{R}^{n\times n}$ be an orthogonal projection matrix, i.e., $P^2 = P$ and $P^T = P$. Suppose $P \neq I$. Let $v \in \mathbb{R}^n$ be a unit vector ($v^T v = 1$) that belongs to the range space of P. Now consider the single input single output system

$$\begin{aligned} \dot{x} &= Px + vu \\ y &= v^T x. \end{aligned}$$

(a) Write the minimal polynomial of P. *Hint:* $P^2 = P$.

(b) Show that $Pv = v$.

(c) Compute e^{Pt}. *Hint: use the power series definition.*

(d) Compute the transfer function $G(s) = Y(s)/U(s)$.

Answer:

a) $P^2 = P \Rightarrow P^2 - P = 0 \Rightarrow P(P-I) = 0$

$\Rightarrow \boxed{m(s) = s(s-1)}$

b) $v \in \text{Range}(P) \Rightarrow \exists x \in \mathbb{R}^n$ s.t. $v = Px$

$\Rightarrow Pv = P(Px) = P^2x = Px = v$ $\blacksquare$

c) $e^{Pt} = I + Pt + P^2\frac{t^2}{2} + P^3\frac{t^3}{3!} + \cdots$

$= I + Pt + P\frac{t^2}{2} + P\frac{t^3}{3!} + \cdots$

$= I + P\underbrace{\left(t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots\right)}_{e^t - 1}$

$\Rightarrow e^{Pt} = I + (e^t - 1)P = \boxed{(I-P) + e^t P}$

d) $G(s) = v^T(sI-P)^{-1}v$

$(sI-P)^{-1} = \mathcal{L}\{e^{Pt}\} = \frac{1}{s}(I-P) + \frac{1}{s-1}P$

$\Rightarrow G(s) = v^T\left\{\frac{1}{s}(I-P) + \frac{1}{s-1}P\right\}v$

$= \frac{1}{s}(v^Tv - v^TPv) + \frac{1}{s-1}v^TPv$

$Pv = v$
$v^Tv = 1$

$= \boxed{\frac{1}{s-1}}$

Q2.

Let $P \in \mathbb{R}^{n\times n}$ be an orthogonal projection matrix, i.e., $P^2 = P$ and $P^T = P$. Suppose $P \neq I$. Let $v \in \mathbb{R}^n$ be a unit vector ($v^T v = 1$) that belongs to the range space of P. Again consider the single input single output system

$$\begin{aligned} \dot{x} &= Px + vu \\ y &= v^T x. \end{aligned}$$

(a) Is this system BIBO stable? Is it internally stable?

(b) Compute the controllable subspace $\mathcal{S}_c$.

(c) Let $w \in \mathbb{R}^n$ be such that w are v are linearly independent. Can we find an input signal $u(t)$ that transfers the state $x(t)$ from the origin $x(0) = 0$ to $x(1) = w$?

(d) Is this system stabilizable?

Answer:

a) $G(s) = \frac{1}{s-1} \Rightarrow \lambda = 1$ is a pole

since $Re\{\lambda\} \geq 0 \Rightarrow$ $\boxed{\text{NOT BIBO stable}}$

$\lambda = 1$ is an eigenvalue of P $(Pv = v)$

since $Re\{\lambda\} > 0 \Rightarrow$ $\boxed{\text{NOT internally stable}}$

b) $S_c = Range[v \;\; Pv \;\; P^2v \;\; \cdots \;\; P^{n-1}v]$ $\quad P^2 = P$

$= Range[v \;\; Pv \;\; Pv \;\; \cdots \;\; Pv]$ $\quad Pv = v$

$= Range[v \;\; v \;\; v \;\; \cdots \;\; v]$

$\Rightarrow \boxed{S_c = span\{v\}}$

c) No, because $w \notin S_c = span\{v\}$

d) $\dim S_c = 1 < n \Rightarrow \underbrace{\text{system uncontrollable}}_{(1)}$

minimal poly. $m(s) = s(s-1) \Rightarrow$ an eigenvalue λ of P can either be $\lambda = 0$ or $\lambda = 1$. That is, $\underbrace{\text{no eigenvalue of } P \text{ satisfies } Re\{\lambda\} < 0.}_{(2)}$

(1) & (2) $\Rightarrow$ $\boxed{\text{NOT stabilizable}}$

Q3.

Consider the system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 5 & -4 \\ 8 & -7 \end{bmatrix} x + \begin{bmatrix} \alpha \\ 1 \end{bmatrix} u \\ y &= [-1 \ \ \beta] x.\end{aligned}$$

(a) Is this system stable in the sense of Lyapunov?

(b) Find α so that the system is BIBO stable for all β.

(c) Is the system stabilizable for the α you found in part (b)?

(d) Find β so that the system is BIBO stable for all α.

Answer:

a) $|sI-A| = \begin{vmatrix} s-5 & 4 \\ -8 & s+7 \end{vmatrix} = (s-5)(s+7)+32$
$= s^2+2s-3$
$= (s+3)(s-1)$

$\Rightarrow \lambda_1 = 1, \ \lambda_2 = -3$

Since $Re\{\lambda_1\} > 0$ $\boxed{\text{system unstable}}$

b) for BIBO stability we need to cancel the pole at $\lambda_1 = 1$.

$TF = C(sI-A)^{-1}B = [-1 \ \ \beta] \dfrac{\begin{bmatrix} s+7 & -4 \\ 8 & s-5 \end{bmatrix}}{(s+3)(s-1)} \begin{bmatrix} \alpha \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} s+7 & -4 \\ 8 & s-5 \end{bmatrix} \begin{bmatrix} \alpha \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha s + 7\alpha - 4 \\ s-5+8\alpha \end{bmatrix} = (s-1)\begin{bmatrix} * \\ * \end{bmatrix}$

$\Rightarrow s-5+8\alpha = s-1 \Rightarrow -5+8\alpha = -1$

$\Rightarrow \boxed{\alpha = 1/2}$

c) Let's find the eigenvector v_1 of A^T for $\lambda_1 = 1$.

$\begin{bmatrix} 5 & 8 \\ -4 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

$\Rightarrow 5a + 8b = a \Rightarrow a + 2b = 0$

$\Rightarrow v_1 = b \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Now check $v_1 \in N(B^T)$?

$B^T v_1 = [\tfrac{1}{2} \ \ 1] \begin{bmatrix} -2b \\ b \end{bmatrix} = 0$

$\Rightarrow$ $\boxed{\text{NOT stabilizable}}$

d) Similar to part (b)

$[-1 \ \ \beta] \begin{bmatrix} s+7 & -4 \\ 8 & s-5 \end{bmatrix} = (s-1)[* \ \ *]$

$\Rightarrow -s-7+8\beta = -(s-1)$

$\Rightarrow -7+8\beta = 1 \Rightarrow \boxed{\beta = 1}$

Q4.

Consider the single input single output system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

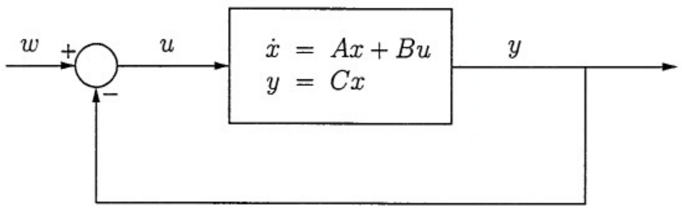
where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -b & -c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(a) Show that this system is controllable.

(b) Show that $\det(sI - A) = s^3 + cs^2 + bs + a$.

For the rest of the question consider the below feedback configuration.



$$\frac{Y(s)}{W(s)} = \frac{\cdots}{(s+1)^3}$$

(c) Find the state space representation of the closed-loop system. I.e., find the triple $(A_{\text{cl}}, B_{\text{cl}}, C_{\text{cl}})$ in terms of A, B, C such that $\dot{x} = A_{\text{cl}}x + B_{\text{cl}}w$ and $y = C_{\text{cl}}x$.

(d) Find C in terms of a, b, c.

a) Construct cont. matrix $[B \;\; AB \;\; A^2B]$

$AB = \begin{bmatrix} 0 \\ 1 \\ -c \end{bmatrix}$, $\quad A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -a & -b & -c \\ * & * & * \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -c \\ * \end{bmatrix}$

$\Rightarrow$ Cont Matrix $= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -c \\ 1 & -c & * \end{bmatrix}$ $\Rightarrow$ this matrix is full row rank

$\Rightarrow$ $\boxed{\text{controllable}}$

b) $|sI - A| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ a & b & s+c \end{vmatrix} = s^2(s+c) + a + bs$

$= s^3 + cs^2 + bs + a$ ☒

c) $\dot{x} = Ax + Bu$
$y = Cx$ $\quad | \quad u = w - y = w - Cx$

$\Rightarrow \dot{x} = Ax + B(w - Cx) = (A - BC)x + Bw$

$y = Cx$

$\Rightarrow$ $\boxed{\begin{aligned} A_{cl} &= A - BC \\ B_{cl} &= B \\ C_{cl} &= C \end{aligned}}$

d) let $C = [k_1 \;\; k_2 \;\; k_3]$

$\Rightarrow A - BC = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a-k_1 & -b-k_2 & -c-k_3 \end{bmatrix}$

$\Rightarrow |sI - A_{cl}| = s^3 + (c+k_3)s^2 + (b+k_2)s + (a+k_1)$

$= (s+1)^3$

$= s^3 + 3s^2 + 3s + 1$

$\Rightarrow k_1 = 1-a, \quad k_2 = 3-b, \quad k_3 = 3-c$

$\Rightarrow$ $\boxed{C = [1-a \quad 3-b \quad 3-c]}$