EE502 First Midterm Exam 20 April 2022

First name:________________________

Last name: KEY

Student ID:________________________

Signature:________________________

Read before you start:

- There are four questions.
- The duration of the examination is 100 minutes.
- The examination is closed-book.
- Please explain all your answers.

Q1	Q2	Q3	Q4	Total

Q1. 20%

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function that satisfies:

- $g(0) = 0$ and $g(y) \neq 0$ for $y \neq 0$,
- $\left[\frac{\partial g}{\partial y}\right]_{y=0} \neq 0$.

Consider the third order system

$$\dddot{y} + g(y) = 0.$$

(5) **(a)** Using a suitable choice of state, obtain the state space representation $\dot{x} = f(x)$.

(5) **(b)** Find the equilibrium point(s) of the system $\dot{x} = f(x)$.

(10) **(c)** Investigate the (local) stability properties of the origin $x = 0$.

a) Let $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$

$$\Rightarrow \left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= \dddot{y} = -g(x_1) \end{aligned}\right\} \quad \boxed{\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} x_2 \\ x_3 \\ -g(x_1) \end{bmatrix}}_{f(x)}}$$

b)

$$f(x) = 0 \Rightarrow \begin{bmatrix} x_2 \\ x_3 \\ -g(x_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_2 = 0$ & $x_3 = 0$ & $-g(x_1) = 0 \Rightarrow x_1 = 0$

Hence, $\boxed{x = 0}$ is the only equilibrium.

c)

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{\partial g}{\partial x_1} & 0 & 0 \end{bmatrix}$$

Let $\alpha^3 = \left.\frac{\partial g}{\partial x_1}\right|_{x_1=0}$. Note that $\alpha \neq 0$

Then the linearization reads

$$\dot{\eta} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha^3 & 0 & 0 \end{bmatrix}}_{A} \eta$$

$$|sI - A| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ \alpha^3 & 0 & s \end{vmatrix} = s^3 + \alpha^3$$

Hence the eigenvalues of A are

$$\lambda_{1,2,3} = -\alpha, \; -\alpha e^{j\frac{2\pi}{3}}, \; -\alpha e^{-j\frac{2\pi}{3}}$$

This means at least one eigenvalue satisfies $Re\{\lambda_i\} > 0$. Therefore the origin $x = 0$ is $\boxed{\text{unstable.}}$

Q2.

30%

Consider the following single-input single-output LTI system. It is known that, when $u = 0$, for each initial condition $x(0)$ the resulting output $y(t)$ is bounded.

$$\dot{x} = \underbrace{\begin{bmatrix} -5 & 3 \\ -6 & 4 \end{bmatrix}}_{A} x + Bu$$

$$y = \underbrace{[c_1 \ -1]}_{C} x.$$

(6) **(a)** Is this system stable in the sense of Lyapunov?

(10) **(b)** Find, if possible, a nonzero $B \in \mathbb{R}^{2\times 1}$ such that the system is not BIBO stable.

(10) **(c)** Find c_1.

(4) **(d)** Find, if possible, a nonzero $B \in \mathbb{R}^{2\times 1}$ such that the impulse response is zero.

a) $|sI - A| = \begin{vmatrix} s+5 & -3 \\ 6 & s-4 \end{vmatrix} = (s-1)(s+2)$

$\Rightarrow \lambda_1 = 1, \ \lambda_2 = -2$

$\Rightarrow$ Due to $\mathrm{Re}\{\lambda_1\} > 0$ the system is $\boxed{\text{unstable.}}$

b) The general form of $y(t)$ (when $u=0$) is $y(t) = \alpha_1 e^{t} + \alpha_2 e^{-2t}$ where α_1, α_2 are determined by $x(0)$. But we're told that $y(t)$ is always bounded. Hence $\alpha_1 = 0$ for all $x(0)$. This implies $y(t) = y(0)e^{-2t}$. Since $y(0) = Cx(0)$ we have:

$$y(t) = e^{-2t} C x(0) \qquad (1)$$

We also have

$$y(t) = Cx(t) = Ce^{At}x(0) \qquad (2)$$

Combining (1) & (2) yields

$$Ce^{At}x(0) = e^{-2t}Cx(0)$$

Now, since $x(0)$ is arbitrary we deduce:

$$Ce^{At} = e^{-2t}C \qquad (3)$$

$$\overset{\mathcal{L}}{\Rightarrow} \ C[sI - A]^{-1} = \frac{1}{s+2} C \qquad (4)$$

The TF reads $H(s) = C[sI-A]^{-1}B$. Hence using (4) we can write

$$H(s) = \frac{CB}{s+2} \qquad (5)$$

$H(s)$ has no unstable poles. Hence the system is $\boxed{\text{BIBO stable}}$ for all B.

c) Differentiate (3) and evaluate at $t=0$.

$$CAe^{At}\Big|_{t=0} = -2e^{-2t}C\Big|_{t=0}$$

$$\Rightarrow CA = -2C \Rightarrow C[A + 2I] = 0$$

$$\Rightarrow [c_1 \ -1]\begin{bmatrix} -3 & 3 \\ -6 & 6 \end{bmatrix} = [0 \ 0]$$

$$\Rightarrow \boxed{c_1 = 2}$$

d) By (5) $CB = 0 \Rightarrow H(s) = 0$. Hence we can choose $\boxed{B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

Q2. 30%

Consider the following single-input single-output LTI system. It is known that, when $u = 0$, for each initial condition $x(0)$ the resulting output $y(t)$ is bounded.

$$\dot{x} = \underbrace{\begin{bmatrix} -5 & 3 \\ -6 & 4 \end{bmatrix}}_{A} x + Bu$$

$$y = \underbrace{[c_1 \ -1]}_{C}\, x\,.$$

(a) Is this system stable in the sense of Lyapunov?

(b) Find, if possible, a nonzero $B \in \mathbb{R}^{2\times 1}$ such that the system is not BIBO stable.

(c) Find c_1.

(d) Find, if possible, a nonzero $B \in \mathbb{R}^{2\times 1}$ such that the impulse response is zero.

Alternative Sol'n

Eigenvalues of A: $\lambda_1 = 1$, $\lambda_2 = -2$

Eigenvectors of A: $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

a) Due to $\mathrm{Re}\{\lambda_1\} > 0$ the system is $\boxed{\text{unstable.}}$

b) $\boxed{\text{Impossible.}}$ Suppose otherwise. Then we can find a B s.t. the impulse response reads

$h(t) = \alpha_1 e^{t} + \alpha_2 e^{-2t}$ with $\alpha_1 \neq 0$. That is,

$h(t)$ is unbounded. Note $h(t) = Ce^{At}B$, i.e., the hmg. response for $x(0) = B$. But every hmg. response is bounded. Contradiction.

c) Hmg. state response has the form

$$x(t) = \beta_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{t} + \beta_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

$y(t) = Cx(t)$ being bounded then implies

$Cv_1 = 0 \implies \boxed{c_1 = 2}$

d) Choose $\boxed{B = v_1}$ Then

$h(t) = Ce^{At}B = Ce^{At}v_1 = Ce^{\lambda_1 t}v_1$

$= e^{\lambda_1 t}(Cv_1) = 0.$

Q3. 30%

Let $\{v_1, v_2, v_3\}$ be an orthonormal basis for $\mathbb{R}^3$, i.e., $v_k^T v_\ell = 0$ for $k \neq \ell$ and $v_k^T v_k = 1$. Furthermore, suppose the vectors v_1, v_2, v_3 have no zero entries. Define $A := v_1 v_2^T - v_2 v_1^T$.

(4) **(a)** Find null (A) and range (A).

(4) **(b)** Find the characteristic polynomial of A. *Hint:* $A^3 = ?$

(3) **(c)** Find the eigenvalues of A.

(4) **(d)** Find the eigenvectors of A.

a) null (A) = span $\{v_3\}$

range (A) = span $\{v_1, v_2\}$

b) $A^2 = (v_1v_2^T - v_2v_1^T)(v_1v_2^T - v_2v_1^T) = -v_1v_1^T - v_2v_2^T$

$A^3 = A^2 \cdot A = -(v_1v_1^T + v_2v_2^T)(v_1v_2^T - v_2v_1^T)$

$= -v_1v_2^T + v_2v_1^T = -A$

$\Rightarrow A^3 + A = 0 \Rightarrow \boxed{d(s) = s(s^2+1)}$

c) Roots of $d(s)$ are the eigenvalues:

$\boxed{\lambda_1 = j, \quad \lambda_2 = -j, \quad \lambda_3 = 0}$

d) $\underline{\lambda_1 = j}$

let $w_1 = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ be the correspon-ding eigenvector.

$Aw_1 = jw_1$

$\Rightarrow [v_1v_2^T - v_2v_1^T](\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3) = j(\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3)$

$\Rightarrow \alpha_2 v_1 - \alpha_1 v_2 = j\alpha_1 v_1 + j\alpha_2 v_2 + j\alpha_3 v_3$

$\Rightarrow \alpha_2 = j\alpha_1, \quad -\alpha_1 = j\alpha_2, \quad 0 = j\alpha_3$

Let $\alpha_1 = 1$. Then $\alpha_2 = j$ and $\alpha_3 = 0$. Hence

$w_1 = v_1 + jv_2$

$\underline{\lambda_2 = -j}$

$w_2 = w_1^* = v_1 - jv_2$

$\underline{\lambda_3 = 0}$

v_3 spans the null space. Hence $w_3 = v_3$.

The eigenvectors of A are therefore:

$$\boxed{\{\underbrace{v_1 + jv_2}_{\lambda_1 = j},\ \underbrace{v_1 - jv_2}_{\lambda_2 = -j},\ \underbrace{v_3}_{\lambda_3 = 0}\}}$$

(3) **(e)** Write the Jordan form of A.

(4) **(f)** Find $\alpha_0(t)$, $\alpha_1(t)$, $\alpha_2(t)$ such that $e^{At} = \alpha_0(t)I + \alpha_1(t)A + \alpha_2(t)A^2$.

(4) **(g)** Determine the stability of the system $\dot{x} = Ax$.

(4) **(h)** Determine the stability of the system $x^+ = Ax$.

e) The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are distinct. Hence:

$$\boxed{J = \begin{bmatrix} j & 0 & 0 \\ 0 & -j & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

f) Using $A^3 = -A$ we can write

$$e^{At} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \frac{t^4}{4!}A^4 + \frac{t^5}{5!}A^5 + \dots$$

$$= I + tA + \frac{t^2}{2}A^2 - \frac{t^3}{3!}A - \frac{t^4}{4!}A^2 + \frac{t^5}{5!}A + \dots$$

$$= I + \underbrace{\left\{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right\}}_{\sin t} A + \underbrace{\left\{\frac{t^2}{2} - \frac{t^4}{4!} + \frac{t^6}{6!} - \dots\right\}}_{\cos t - 1} A^2$$

$$\Rightarrow \boxed{e^{At} = I + \sin t \cdot A + (\cos t - 1)A^2}$$

g) $Re\{\lambda_i\} \leq 0$ & size $(J_i) = 1 \times 1$ $\forall i$

$\Rightarrow$ The system is $\boxed{\text{stable but not exp. stable}}$

h) $|\lambda_i| \leq 1$ & size $(J_i) = 1 \times 1$ $\forall i$

$\Rightarrow$ The system is $\boxed{\text{stable but not exp. stable.}}$

Q4. 20%

Given

$$A = \begin{bmatrix} 2 & -4 \\ -2 & 0 \end{bmatrix}$$

determine (without direct computation) whether a symmetric positive definite solution $P \in \mathbb{R}^{2\times 2}$ exists or not for each of the following matrix equations.

(5) (a) $A^T P + PA = -I$.

(5) (b) $A^T P + PA = I$.

(5) (c) $A^T PA - P = -I$.

(5) (d) $A^T PA - P = I$.

$|sI - A| = (s-2)s - 8 = (s-4)(s+2)$

$\Rightarrow \lambda_1 = 4, \quad \lambda_2 = -2$

Define

$\tilde{A} := -A \Rightarrow \tilde{\lambda}_1 = -4, \quad \tilde{\lambda}_2 = 2$

$\hat{A} := A^{-1} \Rightarrow \hat{\lambda}_1 = \frac{1}{4}, \quad \hat{\lambda}_2 = -\frac{1}{2}$

a) The system $\dot{x} = Ax$ is unstable because $Re\{\lambda_1\} > 0$. Hence the Lyapunov eqn. has $\boxed{\text{NO SOLUTION.}}$

b) $A^T P + PA = I \Leftrightarrow \tilde{A}^T P + P\tilde{A} = -I$

The system $\dot{x} = \tilde{A}x$ is unstable because $Re\{\tilde{\lambda}_2\} > 0$. Hence the Lyapunov eqn. has $\boxed{\text{NO SOLUTION.}}$

c) The system $x^+ = Ax$ is unstable because $|\lambda_1| > 1$. Hence the Lyapunov eqn. has $\boxed{\text{NO SOLUTION.}}$

d) $A^T PA - P = I \Leftrightarrow A^{-T}\{A^T PA - P\}A^{-1} = \underbrace{A^{-T} I A^{-1}}_{Q}$

$\Leftrightarrow P - \hat{A}^T P \hat{A} = Q$

$\Leftrightarrow \hat{A}^T P \hat{A} - P = -Q$

The system $x^+ = \hat{A}x$ is exp. stable because $|\hat{\lambda}_i| < 1$ for $i = 1,2$. Moreover the matrix Q is symmetric pos. def. Hence the $\boxed{\text{SOLUTION EXISTS}}$ for the Lyapunov eqn.