First name:____________________________

Last name:____________________________

Student ID:____________________________

Signature:____________________________

Read before you start:

- There are four questions.
- The duration of the examination is 100 minutes.
- The examination is closed-book.
- Please explain all your answers.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1. 20%

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function that satisfies:

- $g(0) = 0$ and $g(y) \neq 0$ for $y \neq 0$,
- $\left[\frac{\partial g}{\partial y}\right]_{y=0} \neq 0$.

Consider the third order system

$$\dddot{y} + g(y) = 0\,.$$

(a) Using a suitable choice of state, obtain the state space representation $\dot{x} = f(x)$.

(b) Find the equilibrium point(s) of the system $\dot{x} = f(x)$.

(c) Investigate the (local) stability properties of the origin $x = 0$.

Q2. 30%

Consider the following single-input single-output LTI system. It is known that, when $u = 0$, for each initial condition $x(0)$ the resulting output $y(t)$ is bounded.

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -5 & 3 \\ -6 & 4 \end{bmatrix} x + Bu \\ y &= [\, c_1 \;\; -1 \,]\, x\,. \end{aligned}$$

(a) Is this system stable in the sense of Lyapunov?

(b) Find, if possible, a nonzero $B \in \mathbb{R}^{2\times 1}$ such that the system is not BIBO stable.

(c) Find c_1.

(d) Find, if possible, a nonzero $B \in \mathbb{R}^{2\times 1}$ such that the impulse response is zero.

Q3. 30%

Let $\{v_1, v_2, v_3\}$ be an orthonormal basis for $\mathbb{R}^3$, i.e., $v_k^T v_\ell = 0$ for $k \neq \ell$ and $v_k^T v_k = 1$. Furthermore, suppose the vectors v_1, v_2, v_3 have no zero entries. Define $A := v_1 v_2^T - v_2 v_1^T$.

(a) Find $\operatorname{null}(A)$ and $\operatorname{range}(A)$.

(b) Find the characteristic polynomial of A. *Hint:* $A^3 = ?$

(c) Find the eigenvalues of A.

(d) Find the eigenvectors of A.

(e) Write the Jordan form of A.

(f) Find $\alpha_0(t)$, $\alpha_1(t)$, $\alpha_2(t)$ such that $e^{At} = \alpha_0(t)I + \alpha_1(t)A + \alpha_2(t)A^2$.

(g) Determine the stability of the system $\dot{x} = Ax$.

(h) Determine the stability of the system $x^+ = Ax$.

Q4. 20%

Given

$$A = \begin{bmatrix} 2 & -4 \\ -2 & 0 \end{bmatrix}$$

determine (without direct computation) whether a symmetric positive definite solution $P \in \mathbb{R}^{2\times 2}$ exists or not for each of the following matrix equations.

(a) $A^T P + PA = -I$.

(b) $A^T P + PA = I$.

(c) $A^T PA - P = -I$.

(d) $A^T PA - P = I$.