First name: ____________________

Last name: ____________________

Student ID: ____________________

Signature: ____________________

Read before you start:

- There are FIVE QUESTIONS.
- The examination is OPEN-BOOK.
- Besides correctness, the CLARITY of your presentation will also be graded.
- NO COLLABORATION with others.

Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Total**

Q1.

A single-cell organism (say a bacterium) is observed to reach maturity when 3-day old; and it gives birth to one offspring every day once it has reached maturity. The offsprings (and the offsprings of offsprings and so on) themselves evolve in the same manner, i.e., they reach maturity when 3-day old and reproduce at a rate of one offspring per day henceforth. Let y_k be the total number of bacteria in the environment (say within our test tube where we perform the experiment) on the kth day. Suppose we start with a new-born single bacterium on the zeroth day (i.e., $y_0 = 1$). Then we expect to have the following sequence:

$$(y_0,\, y_1,\, y_2,\, y_3,\, y_4,\, y_5,\, y_6,\, y_7,\, y_8,\, y_9,\, y_{10},\, \ldots) = (1,\, 1,\, 1,\, 2,\, 3,\, 4,\, 6,\, 9,\, 13,\, 19,\, 28,\, \ldots)$$

Can this process be represented by a linear model? If your answer is no, explain why not. Otherwise, find matrices A and C such that the above y_k is a solution to

$$\begin{aligned} x_{k+1} &= Ax_k \\ y_k &= Cx_k \end{aligned}$$

for some initial condition x_0. What is your x_0?

Sol'n. Yes. Observe that $y_{k+1} = y_k + y_{k-2}$. Choose, for instance, the state as $x_k = [y_{k-2}\ \ y_{k-1}\ \ y_k]^T$. Then we can write

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} x_k \\ y_k &= [0\ \ 0\ \ 1]x_k \end{aligned}$$

which produces the above sequence for the initial condition $x_0 = [0\ \ 0\ \ 1]^T$.

Q2.

Let $x_{\mathrm{a}}(t)$ and $x_{\mathrm{b}}(t)$ be two solutions of a second-order linear system $\dot{x} = A(t)x$. The initial condition vectors $x_{\mathrm{a}}(0)$ and $x_{\mathrm{b}}(0)$ are linearly independent.

(a) Show that the vectors $x_{\mathrm{a}}(7)$ and $x_{\mathrm{b}}(7)$ must be linearly independent.

(b) Find a closed-form expression for $A(t)$ in terms of the pair $(x_{\mathrm{a}}(\cdot),\, x_{\mathrm{b}}(\cdot))$.

(c) Find a closed-form expression for the state transition matrix $\Phi(t,\, \tau)$ in terms of the pair $(x_{\mathrm{a}}(\cdot),\, x_{\mathrm{b}}(\cdot))$.

Sol'n. (a) Construct the 2×2 matrix $X(t) = [x_{\mathrm{a}}(t) \;\; x_{\mathrm{b}}(t)]$. Since $x_{\mathrm{a}}(0)$ and $x_{\mathrm{b}}(0)$ are linearly independent the matrix $X(0)$ is nonsingular. Note that we can write $X(t) = \Phi(t,\, 0)X(0)$. Now, Φ is always nonsingular and the product of nonsingular matrices is also nonsingular. Therefore $X(t)$ must be nonsingular. This means that its columns $x_{\mathrm{a}}(t)$, $x_{\mathrm{b}}(t)$ remain linearly independent for all t. **(b)** Note that $\dot{X}(t) = A(t)X(t)$. Hence $A(t) = \dot{X}(t)X(t)^{-1}$. **(c)** Note that $X(t) = \Phi(t,\, \tau)X(\tau)$. Hence $\Phi(t,\, \tau) = X(t)X(\tau)^{-1}$.

Q3.

Let $H \in \mathbb{R}^{n\times n}$ denote a matrix with the property $\operatorname{null} H = \operatorname{range} H$.

(a) Find (if possible) an H such that the system $\dot{x} = Hx$ is stable. Explain if impossible.

(b) Find (if possible) an H such that the system $x^+ = Hx$ is unstable. Explain if impossible.

Sol'n. Let $z \in \mathbb{R}^n$ be an arbitrary vector. Define $w = Hz$. Clearly, $w \in \operatorname{range} H$. This allows us to write $w \in \operatorname{null} H$. We then have to have $0 = Hw = H(Hz) = H^2 z$. Since z was arbitrary, it follows that $H^2 = 0$. This at once tells us that all the eigenvalues λ_i of H are at the origin. Therefore we automatically have $|\lambda_i| < 1$ for all i, yielding **(b)** the system $x^+ = Hx$ must be exponentially stable. Note that H cannot be the zero matrix. To see this suppose otherwise, i.e., $H = 0$. Then we would have $\operatorname{null} H = \mathbb{R}^n$ and $\operatorname{range} H = \{0\}$, which would produce the contradiction $\operatorname{null} H \neq \operatorname{range} H$. Now, thanks to $H^2 = 0$, we can write

$$\begin{aligned} e^{Ht} &= I + tH + \frac{t^2}{2}H^2 + \frac{t^3}{3!}H^3 + \frac{t^4}{4!}H^4 + \cdots \\ &= I + tH\,. \end{aligned}$$

Then, since $H \neq 0$, we have $\|e^{Ht}\| \to \infty$ as $t \to \infty$. Consequently, **(a)** the system $\dot{x} = Hx$ must be unstable.

Q4.

Consider the following single-input single-output LTI system of order n

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\,.\end{aligned}$$

Suppose there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ satisfying

$$A^T P + PA + C^T C = 0\,.$$

(a) Show that this system is internally stable.

(b) Show that this system is BIBO stable.

Sol'n. **(a)** Let $x(t)$ be an arbitrary solution of the unforced system $\dot{x} = Ax$. Construct the scalar function $V(t) = x(t)^T Px(t)$. Note that $V(t) \geq \lambda_{\min}(P)\|x(t)\|^2$. We can write

$$\begin{aligned}\dot{V} &= \dot{x}^T Px + x^T P\dot{x} \\ &= x^T A^T Px + x^T PAx \\ &= x^T(A^T P + PA)x \\ &= -x^T C^T Cx \\ &= -y^2\,.\end{aligned}$$

Therefore $\dot{V} \leq 0$, yielding $V(t) \leq V(0)$. This implies $\|x(t)\| \leq \sqrt{V(0)/\lambda_{\min}(P)}$ for all t. In other words, every solution of the unforced system is bounded. Hence the internal stability.

(b) Consider again the unforced system, but this time let $x(t)$ denote the solution starting from the special initial condition $x(0) = B$. That is, $x(t) = e^{At}B$. The corresponding output then is $y(t) = Cx(t) = Ce^{At}B = \mathcal{L}^{-1}\{C(sI - A)^{-1}B\} = \mathcal{L}^{-1}\{G(s)\}$ where $G(s)$ is the transfer function of the original system. In other words, $y(t)$ equals the impulse response $g(t) = \mathcal{L}^{-1}\{G(s)\}$. Hence, by our previous analysis, we can write $g(t)^2 = -\dot{V}(t)$. Integrating both sides we obtain

$$\begin{aligned}\int_0^\infty g(t)^2 dt &= V(0) - \lim_{t\to\infty} V(t) \\ &\leq V(0)\,.\end{aligned}$$

This implies $g(t) \to 0$ as $t \to \infty$, which is only possible if all the poles of $G(s)$ are on the open left half-plane. Hence the BIBO stability.

Q5.

Consider the following second-order nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1^2 - x_1 \\ x_2^2 - x_2 \end{bmatrix}.$$

(a) Find all the equilibrium points.

(b) Determine the (local) stability properties of each equilibrium.

Sol'n. **(a)** The equilibrium points are

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) The Jacobian matrix reads

$$\left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} 0 & 1 - 2x_2 \\ 2 - 4x_1 & 6x_2 - 3 \end{bmatrix}$$

yielding the characteristic polynomial

$$d(s) = s^2 + [3 - 6x_2]s + [(4x_1 - 2)(1 - 2x_2)].$$

By evaluating the roots of $d(s)$ at the equilibrium points we reach the following conclusion.

- $x_{\text{eq}} = [0 \;\; 0]^T$: $d(s) = s^2 + 3s - 2 \implies \lambda_{1,2} = 0.5616, -3.5616$. There is an eigenvalue on the open right half-plane. Hence unstable.
- $x_{\text{eq}} = [1 \;\; 0]^T$: $d(s) = s^2 + 3s + 2 \implies \lambda_{1,2} = -1, -2$. All the eigenvalues are on the open left half-plane. Hence stable.
- $x_{\text{eq}} = [0 \;\; 1]^T$: $d(s) = s^2 - 3s + 2 \implies \lambda_{1,2} = 1, 2$. There is an eigenvalue on the open right half-plane. Hence unstable.
- $x_{\text{eq}} = [1 \;\; 1]^T$: $d(s) = s^2 - 3s - 2 \implies \lambda_{1,2} = -0.5616, 3.5616$. There is an eigenvalue on the open right half-plane. Hence unstable.