First name:______________________________

Last name:______________________________

Student ID:______________________________

Signature:______________________________

Read before you start:

- There are FIVE QUESTIONS.
- The examination is OPEN-BOOK.
- Besides correctness, the CLARITY of your presentation will also be graded.
- NO COLLABORATION with others.

Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Total**

Q1.

A single-cell organism (say a bacterium) is observed to reach maturity when 3-day old; and it gives birth to one offspring every day once it has reached maturity. The offsprings (and the offsprings of offsprings and so on) themselves evolve in the same manner, i.e., they reach maturity when 3-day old and reproduce at a rate of one offspring per day henceforth. Let y_k be the total number of bacteria in the environment (say within our test tube where we perform the experiment) on the kth day. Suppose we start with a new-born single bacterium on the zeroth day (i.e., $y_0 = 1$). Then we expect to have the following sequence:

$$(y_0,\, y_1,\, y_2,\, y_3,\, y_4,\, y_5,\, y_6,\, y_7,\, y_8,\, y_9,\, y_{10},\, \ldots) = (1,\, 1,\, 1,\, 2,\, 3,\, 4,\, 6,\, 9,\, 13,\, 19,\, 28,\, \ldots)$$

Can this process be represented by a linear model? If your answer is no, explain why not. Otherwise, find matrices A and C such that the above y_k is a solution to

$$\begin{aligned} x_{k+1} &= Ax_k \\ y_k &= Cx_k \end{aligned}$$

for some initial condition x_0. What is your x_0?

Q2.

Let $x_{\mathrm{a}}(t)$ and $x_{\mathrm{b}}(t)$ be two solutions of a second-order linear system $\dot{x} = A(t)x$. The initial condition vectors $x_{\mathrm{a}}(0)$ and $x_{\mathrm{b}}(0)$ are linearly independent.

(a) Show that the vectors $x_{\mathrm{a}}(7)$ and $x_{\mathrm{b}}(7)$ must be linearly independent.

(b) Find a closed-form expression for $A(t)$ in terms of the pair $(x_{\mathrm{a}}(\cdot),\, x_{\mathrm{b}}(\cdot))$.

(c) Find a closed-form expression for the state transition matrix $\Phi(t,\, \tau)$ in terms of the pair $(x_{\mathrm{a}}(\cdot),\, x_{\mathrm{b}}(\cdot))$.

Q3.

Let $H \in \mathbb{R}^{n\times n}$ denote a matrix with the property $\operatorname{null} H = \operatorname{range} H$.

(a) Find (if possible) an H such that the system $\dot{x} = Hx$ is stable. Explain if impossible.

(b) Find (if possible) an H such that the system $x^+ = Hx$ is unstable. Explain if impossible.

Q4.

Consider the following single-input single-output LTI system of order n

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\,.\end{aligned}$$

Suppose there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ satisfying

$$A^T P + PA + C^T C = 0\,.$$

(a) Show that this system is internally stable.

(b) Show that this system is BIBO stable.

Q5.

Consider the following second-order nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1^2 - x_1 \\ x_2^2 - x_2 \end{bmatrix}.$$

(a) Find all the equilibrium points.

(b) Determine the (local) stability properties of each equilibrium.