First name:__________________________

Last name: KEY

Student ID:__________________________

Signature:__________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 100 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS.

Q1	Q2	Q3	Q4	Total

Q1.

Consider the first-order LTV system

$$\dot{x} = A(t)x\,, \qquad t \geq 0$$

a solution of which is known to be $x(t) = \dfrac{1}{1+t}$.

(6) **(a)** Find $A(t)$.

(6) **(b)** Find the state transition matrix $\Phi(t,\, t_0)$.

(6) **(c)** Find the solution starting from the initial condition $x(3) = -7$.

(7) **(d)** Is this system stable? Is it asymptotically stable? Is it exponentially stable?

a) $A(t) = \dfrac{\dot{x}(t)}{x(t)} = \dfrac{-\frac{1}{(1+t)^2}}{\frac{1}{1+t}} = \boxed{-\dfrac{1}{1+t}}$

b) $x(t) = \Phi(t,0)\,x(0) \Rightarrow \Phi(t,0) = \dfrac{x(t)}{x(0)} = \dfrac{1}{1+t}$

$\Phi(t,t_0) = \Phi(t,0)\,\Phi(0,t_0) = \Phi(t,0)\,\Phi(t_0,0)^{-1} = \boxed{\dfrac{1+t_0}{1+t}}$

c) $x(t) = \Phi(t,3)\,x(3) = \dfrac{1+3}{1+t}\cdot(-7) = \boxed{\dfrac{-28}{1+t}}$

d) $x(t) = \dfrac{1+t_0}{1+t}\,x(t_0) \Rightarrow |x(t)| = \dfrac{1+t_0}{1+t}\,|x(t_0)| \quad (1)$

$(1) \Rightarrow |x(t)| \leq |x(t_0)|$ for $t \geq t_0 \Rightarrow$ $\boxed{\text{stable}}$

$(1) \Rightarrow \lim_{t\to\infty} |x(t)| = 0 \Rightarrow$ $\boxed{\text{asy. stable}}$

The sol'n $x(t) = \dfrac{1}{1+t}$ cannot be upperbounded by an exponential decaying funct. $ce^{-\lambda t}$.

$\Rightarrow$ $\boxed{\text{NOT exp. stable}}$

Q2.

The impulse response of a single input single output LTI system reads $h(t) = 1 + te^{-3t}$.

(5) **(a)** Obtain the transfer function $H(s)$ of this system.

(10) **(b)** Write a state space representation $\left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array} \right\}$ for this system.

(5) **(c)** Write the Jordan form of A.

(5) **(d)** Find an initial condition $x(0)$ which yields $y(t) = h(t)$ for zero input $u(t) \equiv 0$.

a) $\mathcal{L}\{h(t)\} = \frac{1}{s} + \frac{1}{(s+3)^2} = \boxed{\frac{s^2+7s+9}{s^3+6s^2+9s}}$

b) let $X_3 := \frac{1}{s^3+6s^2+9s} U$

$X_2 := sX_3 \qquad (1)$

$X_1 := sX_2 = s^2 X_3 \qquad (2)$

$sX_1 = s^3 X_3 = -6s^2X_3 - 9sX_3 + U \qquad (3)$

$Y = s^2X_3 + 7sX_3 + 9X_3 \qquad (4)$

(1),(2),(3) $\Rightarrow$ $\dot{x} = \begin{bmatrix} -6 & -9 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$

(4) $\Rightarrow$ $y = \begin{bmatrix} 1 & 7 & 9 \end{bmatrix} x$

c) $\boxed{J = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$

d) $h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{C[sI-A]^{-1}B\}$

$= C\mathcal{L}^{-1}\{[sI-A]^{-1}\}B = Ce^{At}B \quad (*)$

$y(t) = Ce^{At}x(0) \quad (**)$

$(*)$ & $(**)$ $\Rightarrow$ $x(0) = B = \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$

Q3.

Let a matrix $A \in \mathbb{R}^{n\times n}$ satisfy

$$A^T P A - \rho^2 P < 0$$

for some symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and positive real number ρ.

(15) (a) Show that all eigenvalues of A satisfy $|\lambda_i| < \rho$.

(10) (b) Let $x(k)$ denote the solution of the system $x^+ = Ax$. Show that we can find some $c > 0$ such that $\|x(k)\| \leq c\rho^k \|x(0)\|$ for $k = 0, 1, 2, \ldots$

Let $\tilde{A} := \frac{1}{\rho} A \Rightarrow \tilde{\lambda}_i = \frac{1}{\rho}\lambda_i$

& $\tilde{P} := \rho^2 P \Rightarrow \tilde{P} = \tilde{P}^T > 0$

a) $\tilde{A}^T \tilde{P} \tilde{A} - \tilde{P} = \left(\frac{A}{\rho}\right)^T (\rho^2 P)\left(\frac{A}{\rho}\right) - \rho^2 P$

$= A^T P A - \rho^2 P < 0 \quad (1)$

Lyap. stability test $\Rightarrow |\tilde{\lambda}_i| < 1 \Rightarrow |\lambda_i| < \rho$ ☒

b) Consider the sys. $\tilde{x}^+ = \tilde{A}\tilde{x}$. Set $\tilde{x}(0) = x(0)$

$\Rightarrow \tilde{x}(k) = \tilde{A}^k \tilde{x}(0) = \left(\frac{A}{\rho}\right)^k x(0) = \frac{1}{\rho^k} A^k x(0) = \frac{x(k)}{\rho^k}$

sys. $\tilde{x}^+ = \tilde{A}\tilde{x}$ is exp. stable by (1)

$\Rightarrow \|\tilde{x}(k)\| \leq c\alpha^k \|\tilde{x}(0)\|$ for $c > 0$, $0 < \alpha < 1$, $k \geq 0$

$\Rightarrow \|\tilde{x}(k)\| \leq c \|\tilde{x}(0)\|$ for all k

$\Rightarrow \frac{\|x(k)\|}{\rho^k} \leq c \|x(0)\| \Rightarrow \|x(k)\| \leq c\rho^k \|x(0)\|$ ☒

Q4.

Consider

$$\begin{aligned}\dot{x} &= \begin{bmatrix} -6 & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x \\ y &= [4 \;\; c_2]\, x\,.\end{aligned}$$

It is known that $[1 \;\; 2]^T$ is an equilibrium point for this system. Moreover, it has been observed that for all initial conditions $x(0) = x_0$ the output always has the form $y(t) = ke^{-t}$, where the constant k depends on x_0.

(10) **(a)** Find the eigenvectors of the system matrix A.

(5) **(b)** Determine null A and range A.

(5) **(c)** Find c_2.

(5) **(d)** Determine the stability properties of the system.

$\bar{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ eq. point $\Rightarrow A\bar{x} = 0$

$\Rightarrow \begin{bmatrix} -6 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow a_{12} = 3$ & $a_{21} = -2a_{22}$

$y(t) = ke^{-t} \Rightarrow \lambda_1 = -1$ is an eigenvalue

$A = \begin{bmatrix} -6 & 3 \\ -2a_{22} & a_{22} \end{bmatrix} \Rightarrow |sI - A| = \begin{vmatrix} s+6 & -3 \\ 2a_{22} & s-a_{22} \end{vmatrix}$

$\Rightarrow |sI - A| = (s+6)(s-a_{22}) + 6a_{22} = s(s+6-a_{22})$

$|sI-A|_{s=-1} = 0 \Rightarrow a_{22} = 5$

$\Rightarrow A = \begin{bmatrix} -6 & 3 \\ -10 & 5 \end{bmatrix}$

a) $\underline{\lambda_1 = -1}$ $\quad [A + I]v_1 = 0 \Rightarrow \begin{bmatrix} -5 & 3 \\ -10 & 6 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \boxed{v_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}}$

$\underline{\lambda_2 = 0}$ $\quad \boxed{v_2 = \bar{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

b) null A = span $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

range A = span $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

c) λ_2 doesn't show up in $y(t)$. Therefore

$v_2 \in$ null C

$\Rightarrow [4 \;\; c_2]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0 \Rightarrow \boxed{c_2 = -2}$

d) Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 0$

order: 2

$\Rightarrow$ stable but not exp. stable