First name:________________

Last name: KEY

Student ID:________________

Signature:________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 100 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS.

Q1	Q2	Q3	Q4	Total

Q1.

Let $M \in \mathbb{R}^{10\times 10}$ be the multiplication table, i.e., $m_{ij} = i \times j$. Note that we can write $M = [1\ 2\ 3\ \cdots\ 10]^T \times [1\ 2\ 3\ \cdots\ 10]$.

(a) Write the range space of M.

(b) Find the eigenvalues of M.

(c) Write the Jordan form of M.

(d) Discuss the stability of the system $\dot{x} = -Mx$.

a) Let $z = [1\ 2\ \dots\ 10]^T$

$M = zz^T \Rightarrow \boxed{\text{range}(M) = \text{span}\{z\}}$

b) $\text{rank}(M) = 1 \Rightarrow$ M has (at least) $10 - 1 = 9$ eigenvalues at the origin.

Also, $Mz = zz^T z = \|z\|^2 z$

$\Rightarrow \|z\|^2 = 385$ is another eigenvalue

$\Rightarrow \boxed{\{\lambda_1, \lambda_2, \dots, \lambda_9, \lambda_{10}\} = \{0, 0, \dots, 0, 385\}}$

c) M is symmetric hence its Jordan form J is diagonal $\boxed{J = \text{diag}(0, 0, \dots, 0, 385)}$

d) Jordan form of $-M$ is $-J$. Hence,

$$\left.\begin{array}{l} \lambda_i(-M) \le 0 \text{ for all } i,\ \lambda_{1,2,\dots,9} = 0 \\ \text{\& all the Jordan blocks are size } 1\times 1 \end{array}\right\} (*)$$

$(*) \Rightarrow \dot{x} = -Mx$ is $\boxed{\text{stable but not exp. stable}}$

Q2.

Let $L \in \mathbb{R}^{n\times n}$ $(n \geq 137)$ be the normalized Laplacian matrix of a complete graph. That is, $L = I - \frac{1}{n}bb^T$, where $b \in \mathbb{R}^n$ is the vector of ones, i.e., $b = [1\ 1\ \cdots\ 1]^T$.

(a) Show that L is a projection matrix, i.e., $L^T = L$ and $L^2 = L$.

(b) Is L positive (semi)definite? *Hint:* $Lb = ?$

(c) Compute e^{Lt}.

(d) Let $v = [v_1\ v_2\ \cdots\ v_n]^T$ denote the vector of voltages of identical capacitors coupled via identical resistors. Then the normalized dynamics of this bank of capacitors read $\dot{v} = -Lv$. Find the limit $\lim_{t\to\infty} v_i(t)$ if it exists.

a) $L^T = (I - \frac{1}{n}bb^T)^T = I - \frac{1}{n}bb^T = L$

$L^2 = (I - \frac{1}{n}bb^T)^2 = I - \frac{2}{n}bb^T + \frac{1}{n^2}b\underbrace{b^Tb}_{n}b^T$

$= I - \frac{2}{n}bb^T + \frac{1}{n}bb^T = I - \frac{1}{n}bb^T = L$ ∎

b) Let $x \in \mathbb{R}^n$ be arbitrary.

$x^TLx = x^TL^2x = x^TL^TLx = \|Lx\|^2 \geq 0$

$\Rightarrow$ $\boxed{L \text{ is pos. semidef.}}$

How about pos. def.?

$Lb = (I - \frac{1}{n}bb^T)b = b - \frac{1}{n}b\underbrace{b^Tb}_{n} = b - b = 0$

$\Rightarrow$ L has an eigenvalue at the origin

$\Rightarrow$ $\boxed{L \text{ cannot be pos. def.}}$

c) $e^{Lt} = I + Lt + \frac{L^2t^2}{2} + \frac{L^3t^3}{3!} + \dots$ $\quad L^k = L$

$= I + Lt + L\frac{t^2}{2} + L\frac{t^3}{3!} + \dots$

$= I + L\underbrace{\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots\right)}_{e^t - 1}$

$\Rightarrow$ $\boxed{e^{Lt} = I + (e^t - 1)L}$

d) $e^{-Lt} = e^{L(-t)} = I + (e^{-t} - 1)L$

$\Rightarrow \lim_{t\to\infty} e^{-Lt} = I - L = \frac{1}{n}bb^T$

$\Rightarrow \lim_{t\to\infty} v(t) = \lim_{t\to\infty} e^{-Lt}v(0) = \frac{1}{n}bb^Tv(0)$

$\Rightarrow$ $\boxed{\lim_{t\to\infty} v_i(t) = \frac{v_1(0) + v_2(0) + \dots + v_n(0)}{n}}$

Hence, all the node voltages converge to the mean of the initial node voltages.

Q3.

Given:

- $W \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix satisfying $W < I$.
- $S \in \mathbb{R}^{n \times n}$ is a skew-symmetric matrix, i.e., $S^T + S = 0$.
- $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, i.e., $Q^T Q = I$.

Determine the stability of the below systems. *Your solution should be general. That is, do not assign numerical values to the parameters n, W, S, Q.*

(a) $\dot{x} = [S - W]x$.

(b) $\dot{x} = [S + W]x$.

(c) $x^+ = QWx$.

(d) $x^+ = WQx$.

Lyapunov eqn. for exp stability :

$A^T P + PA < 0$ for $\dot{x} = Ax$

$A^T PA - P < 0$ for $x^+ = Ax$

a) $A = [S - W]$. Choose $P = I$

$A^T P + PA = [S-W]^T + [S-W]$

$= \underbrace{S^T + S}_{0} - W^T - W = -2W < 0$

$\Rightarrow$ $\boxed{\text{exp. stable}}$

b) $A = [S + W]$. Choose $P = I$

$A^T P + PA = 2W > 0$ (inconclusive)

let $\tilde{A} := -A$

$\tilde{A}^T P + P\tilde{A} = -2W < 0$

$\Rightarrow$ $\dot{x} = \tilde{A}x$ is exp. stable

$\Rightarrow$ $Re\{\lambda_i(\tilde{A})\} < 0$ for all i $\quad \lambda_i(A) = -\lambda_i(\tilde{A})$

$\Rightarrow$ $Re\{\lambda_i(A)\} > 0$ for all i

$\Rightarrow$ $\boxed{\text{unstable}}$

c) $A = QW$. Choose $P = I$

$A^T PA - P = W\underbrace{Q^T Q}_{I} W - I = W^2 - I$

$W^2 - I$ is neg. def. if all its eigenvalues satisfy $\lambda_i(W^2 - I) < 0$.

$W - I < 0 \Rightarrow \lambda_i(W - I) < 0 \Rightarrow \lambda_i(W) - 1 < 0$

$\left.\begin{array}{l} \Rightarrow \lambda_i(W) < 1 \\ \& \ \lambda_i(W) > 0 \end{array}\right\} \lambda_i(W^2) = (\lambda_i(W))^2 < 1$

$\Rightarrow \lambda_i(W^2) - 1 < 0 \Rightarrow \lambda_i(W^2 - I) < 0$

$\Rightarrow$ $\boxed{\text{exp. stable}}$

d) $A = WQ$. Choose $P = W^{-2}$

$A^T PA - P = Q^T W W^{-2} W Q - W^{-2} = I - W^{-2}$

$\lambda_i(I - W^{-2}) = 1 - \lambda_i(W^2)^{-1} = \dfrac{\lambda_i(W^2) - 1}{\lambda_i(W^2)} < 0$

$\Rightarrow$ $\boxed{\text{exp. stable}}$

Q4.

Consider the second-order system below.

$$\dot{x} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} x$$

(a) Compute the eigenvalues and the eigenvectors of the system matrix.

(b) Let $u(t) \equiv 0$. Find (if possible) a nonzero initial condition $x(0)$ so that $y(t)$ is bounded.

(c) Let $u(t) \equiv 0$. Find (if possible) a nonzero initial condition $x(0)$ so that $x(t)$ is bounded.

(d) Let $u(t) = 12e^{-t}$. Find (if possible) an initial condition $x(0)$ so that $x(t)$ is bounded.

a) $|sI - A| = \begin{vmatrix} s-4 & 1 \\ -2 & s-1 \end{vmatrix} = (s-4)(s-1) + 2 = s^2 - 5s + 6$

$= (s-2)(s-3) \Rightarrow \boxed{\lambda_1 = 2,\ \lambda_2 = 3}$

$[\lambda_1 I - A]v_1 = 0 \Rightarrow \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (1)

$[\lambda_2 I - A]v_2 = 0 \Rightarrow \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (2)

(1) & (2) $\Rightarrow \boxed{v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

b) Note that $Cv_2 = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

choose $x(0) = v_2 \Rightarrow x(t) = v_2 e^{\lambda_2 t}$

$\Rightarrow y(t) = Cv_2 e^{\lambda_2 t} = 0$

$\Rightarrow \boxed{x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

c) Impossible because $Re\{\lambda_i\} > 0$ for $i = 1, 2$.

d) For $u(t) = 12e^{-t}$ the form of the solution:

$x(t) = \alpha v_1 e^{\lambda_1 t} + \beta v_2 e^{\lambda_2 t} + v_3 e^{-t}$

boundedness $\Rightarrow \alpha = 0,\ \beta = 0$

$\Rightarrow x(t) = v_3 e^{-t} \Rightarrow x(0) = v_3$

$v_3 = ?$

$\dot{x} = Ax + Bu \Rightarrow \frac{d}{dt}\{v_3 e^{-t}\} = Av_3 e^{-t} + B \cdot 12e^{-t}$

$\Rightarrow -v_3 e^{-t} = Av_3 e^{-t} + 12B e^{-t}$

$\Rightarrow -v_3 = Av_3 + 12B$

$\Rightarrow [-I - A]v_3 = 12B$

$\Rightarrow v_3 = 12[-I - A]^{-1}B$

$= 12 \begin{bmatrix} -5 & 1 \\ -2 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$

$\Rightarrow \boxed{x(0) = \begin{bmatrix} -1 \\ -5 \end{bmatrix}}$