First name:____________________________

Last name:____________________________

Student ID:____________________________

Signature:____________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 100 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

Let $M \in \mathbb{R}^{10\times 10}$ be the multiplication table, i.e., $m_{ij} = i \times j$. Note that we can write $M = [1\ 2\ 3\ \cdots\ 10]^T \times [1\ 2\ 3\ \cdots\ 10]$.

(a) Write the range space of M.

(b) Find the eigenvalues of M.

(c) Write the Jordan form of M.

(d) Discuss the stability of the system $\dot{x} = -Mx$.

Q2.

Let $L \in \mathbb{R}^{n \times n}$ $(n \geq 137)$ be the normalized Laplacian matrix of a complete graph. That is, $L = I - \frac{1}{n} b b^T$, where $b \in \mathbb{R}^n$ is the vector of ones, i.e., $b = [1\ 1\ \cdots\ 1]^T$.

(a) Show that L is a projection matrix, i.e., $L^T = L$ and $L^2 = L$.

(b) Is L positive (semi)definite? *Hint:* $Lb = ?$

(c) Compute e^{Lt}.

(d) Let $v = [v_1\ v_2\ \cdots\ v_n]^T$ denote the vector of voltages of identical capacitors coupled via identical resistors. Then the normalized dynamics of this bank of capacitors read $\dot{v} = -Lv$. Find the limit $\lim_{t \to \infty} v_i(t)$ if it exists.

Q3.

Given:

- $W \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix satisfying $W < I$.
- $S \in \mathbb{R}^{n \times n}$ is a skew-symmetric matrix, i.e., $S^T + S = 0$.
- $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, i.e., $Q^T Q = I$.

Determine the stability of the below systems. *Your solution should be general. That is, do not assign numerical values to the parameters n, W, S, Q.*

(a) $\dot{x} = [S - W]x$.

(b) $\dot{x} = [S + W]x$.

(c) $x^+ = QWx$.

(d) $x^+ = WQx$.

Q4.

Consider the second-order system below.

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & -1 \end{bmatrix} x \end{aligned}$$

(a) Compute the eigenvalues and the eigenvectors of the system matrix.

(b) Let $u(t) \equiv 0$. Find (if possible) a nonzero initial condition $x(0)$ so that $y(t)$ is bounded.

(c) Let $u(t) \equiv 0$. Find (if possible) a nonzero initial condition $x(0)$ so that $x(t)$ is bounded.

(d) Let $u(t) = 12e^{-t}$. Find (if possible) an initial condition $x(0)$ so that $x(t)$ is bounded.