First name:________________________

Last name: KEY

Student ID:________________________

Did you take EE501? ☐ Yes ☐ No

Signature:________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 100 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS.

Q1	Q2	Q3	Q4	Total

Q1.

Given two nonzero vectors B, $C^T \in \mathbb{R}^n$ consider the system

$$\begin{aligned} \dot{x} &= BCx + Bu \\ y &= Cx. \end{aligned}$$

(a) Obtain the state transition matrix. Characterize stability in terms of $\lambda := CB$.

(b) Is this system zero state equivalent to $\left\{ \begin{aligned} \dot{z} &= CBz + CBu \\ y &= z \end{aligned} \right\}$?

a)

$$e^{BCt} = I + BCt + (BC)^2\frac{t^2}{2} + (BC)^3\frac{t^3}{3!} + \cdots$$

$$= I + BCt + B\underbrace{CB}_{\lambda}C\frac{t^2}{2} + B\underbrace{CBCB}_{\lambda^2}C\frac{t^3}{3!} + \cdots$$

$$= I + BCt + BC\frac{\lambda t^2}{2} + BC\frac{\lambda^2 t^3}{3!} + \cdots \qquad (1)$$

If $\lambda = 0$ then (1) $\Rightarrow$ $e^{BCt} = I + BCt$

If $\lambda \neq 0$, $e^{BCt} = I + \frac{BC}{\lambda}\underbrace{\left(\lambda t + \frac{\lambda^2 t^2}{2} + \frac{\lambda^3 t^3}{3!} + \cdots\right)}_{e^{\lambda t} - 1}$

Hence,

$$\boxed{e^{BCt} = \begin{cases} I + BCt & \text{for } \lambda = 0 \\ I + \dfrac{e^{\lambda t} - 1}{\lambda} BC & \text{for } \lambda \neq 0 \end{cases}}$$

Looking at e^{BCt} we can say

$\lambda \geq 0 \Rightarrow$ system unstable

$\lambda < 0 \Rightarrow$ system stable but not exp. stable (unless $n=1$)

b) YES. For the second system we can write

$$\dot{y} = CBy + CBu = \lambda y + \lambda u$$

$$\Rightarrow sY = \lambda Y + \lambda U \Rightarrow \frac{Y}{U} = \boxed{\frac{\lambda}{s-\lambda} = G_2(s)}$$

For the first system

$$\dot{y} = C\dot{x} = C\{BCx + Bu\}$$

$$= CBCx + CBu = \lambda Cx + \lambda u$$

$$= \lambda y + \lambda u \Rightarrow \boxed{G_1(s) = \frac{\lambda}{s-\lambda}} = G_2(s)$$

Another way to obtain $G_1(s)$ is to use e^{BCt}:

$$G_1(s) = C(sI - BC)^{-1}B$$

$$= C\mathcal{L}\{e^{BCt}\}B = C\mathcal{L}\left\{I + \frac{e^{\lambda t}-1}{\lambda}BC\right\}B$$

$$= C\left\{\frac{1}{s}I + \left(\frac{1}{s-\lambda} - \frac{1}{s}\right)\frac{BC}{\lambda}\right\}B$$

$$= \frac{\lambda}{s} + \lambda\left(\frac{1}{s-\lambda} - \frac{1}{s}\right) = \boxed{\frac{\lambda}{s-\lambda}}$$ as expected.

Q2.

For each of the below cases determine the stability of the system $x^+ = Ax$.

(a) $A^T A - I < 0$.

(b) $A^T A - I = 0$.

(c) $A^T A - I > 0$.

a) Let $P = I$ (which satisfies $P = P^T > 0$)

$\Rightarrow A^T P A - P = A^T A - I < 0$

By Lyapunov Thm. system is $\boxed{\text{exp. stable}}$

b) Let $V(x) := x^T x = \|x\|^2$

$\Rightarrow V(x^+) - V(x) = (Ax)^T (Ax) - x^T x$

$= x^T A^T A x - x^T x = x^T (A^T A - I) x$

$= 0$

$\Rightarrow \|x(k+1)\| = \|x(k)\| \Rightarrow \|x(k)\| = \|x(0)\|$ for all k.

$\Rightarrow$ $\boxed{\text{stable but not exp. stable}}$

c) Let $\tilde{A} := A^{-1}$

$\Rightarrow \tilde{A}^T \tilde{A} - I = \tilde{A}^T \underbrace{(I - A^T A)}_{<0} \tilde{A} < 0$

$\Rightarrow$ By Lyapunov Thm. eigenvalues of $\tilde{A}$ satisfy $|\tilde{\lambda}_i| < 1$

$\Rightarrow$ Eigenvalues of A satisfy $|\lambda_i| = |\tilde{\lambda}_i^{-1}| > 1$

$\Rightarrow$ $\boxed{\text{unstable}}$

Q3.

For $A, B \in \mathbb{R}^{n\times n}$ consider the LTV system

$$\dot{x} = e^{-At} B e^{At} x.$$

(a) Obtain the state transition matrix $\Phi(t, \tau)$. *Hint: employ* $z(t) := e^{At}x(t)$.

(b) Find the solution $x(t)$ given that $Ax(0) = \alpha x(0)$ and $Bx(0) = \beta x(0)$ for some $\alpha, \beta \in \mathbb{R}$.

a) $\dot{z} = \frac{d}{dt}\{e^{At}x\} = \dot{\widehat{e^{At}}}\, x + e^{At}\dot{x}$

$= Ae^{At}x + e^{At}\{e^{-At}B e^{At}\}x$

$= Ae^{At}x + Be^{At}x = (A+B)e^{At}x$

$= (A+B)z$

$\Rightarrow z(t) = e^{(A+B)(t-t_0)} z(t_0)$

Now, $x(t) = e^{-At} z(t)$

$= e^{-At} e^{(A+B)(t-t_0)} z(t_0)$

$= \underbrace{e^{-At} e^{(A+B)(t-t_0)} e^{At_0}}_{\Phi(t,t_0)} x(t_0)$

$\Rightarrow \boxed{\Phi(t,\tau) = e^{-At} e^{(A+B)(t-\tau)} e^{A\tau}}$

b) $\left.\begin{array}{l} Ax(0) = \alpha x(0) \\ Bx(0) = \beta x(0) \end{array}\right\} \Rightarrow$ $x(0)$ is an eigenvector of both A & B.

$\Rightarrow (A+B)x(0) = (\alpha+\beta)x(0) \Rightarrow$ $x(0)$ is an eigenvector of $A+B$

$\Rightarrow$ $x(0)$ is also an eigenvector of e^{-At}, $e^{(A+B)t}$ with eigenvalues $e^{-\alpha t}$, $e^{(\alpha+\beta)t}$, resp.

Now, $x(t) = \Phi(t,0)x(0)$

$= e^{-At} e^{(A+B)t} x(0)$

$= e^{-At}(e^{(\alpha+\beta)t} x(0))$

$= e^{-\alpha t} e^{(\alpha+\beta)t} x(0) = \boxed{e^{\beta t} x(0)}$

Q4.

For $y \in \mathbb{R}^n$ and $Q = Q^T > 0$ consider the system

$$\ddot{y} + Q^{-1} y = 0.$$

(a) Obtain a state space representation $\left\{ \begin{array}{rcl} \dot{x} & = & Ax \\ y & = & Cx \end{array} \right\}$.

(b) Find a block diagonal $P = P^T > 0$ that satisfies $A^T P + PA = 0$.

(c) Can A have a 2×2 Jordan block?

a) Let $x_1 := y$ & $x_2 := \dot{y}$

$$\Rightarrow \dot{x}_1 = \dot{y} = x_2$$

$$\dot{x}_2 = \ddot{y} = -Q^{-1} y = -Q^{-1} x_1$$

Hence,
$$\boxed{\dot{x} = \begin{bmatrix} 0 & I \\ -Q^{-1} & 0 \end{bmatrix} x \quad , \quad y = \begin{bmatrix} I & 0 \end{bmatrix} x}$$

b) Let $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$

$$A^T P + PA = \begin{bmatrix} 0 & -Q^{-1} \\ I & 0 \end{bmatrix}\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} + \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\begin{bmatrix} 0 & I \\ -Q^{-1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -Q^{-1}P_2 \\ P_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & P_1 \\ -P_2 Q^{-1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -Q^{-1}P_2 + P_1 \\ P_1 - P_2 Q^{-1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} P_1 - P_2 Q^{-1} = 0 \\ P_1 - Q^{-1} P_2 = 0 \end{array}\right\} \text{ choose } P_1 = I \text{ \& } P_2 = Q$$

$$\boxed{\Rightarrow P = \begin{bmatrix} I & 0 \\ 0 & Q \end{bmatrix}}$$

c) NO. Let $V(x) := x^T P x$

$$\Rightarrow \dot{V} = \dot{x}^T P x + x^T P \dot{x} = x^T (A^T P + PA) x = 0$$

$$\Rightarrow V(x(t)) = V(x(0))$$

Then

$$\|x(t)\|^2 \leq \frac{1}{\lambda_{min}(P)} V(x(t)) = \frac{V(x(0))}{\lambda_{min}(P)} \quad \text{for all } t.$$

$\Rightarrow \|x(t)\| \not\to \infty \Rightarrow$ system $\dot{x} = Ax$ is stable.

$\Rightarrow \operatorname{Re}\{\lambda_i\} \leq 0 \quad (1)$

Also, $(-A)^T P + P(-A) = 0 \Rightarrow \dot{x} = -Ax$ is stable

$\Rightarrow \operatorname{Re}\{-\lambda_i\} \leq 0 \quad (2)$

(1)&(2) $\Rightarrow \operatorname{Re}\{\lambda_i\} = 0$ for all i. Since system is stable this means all the Jordan blocks are of size 1×1. ∎