First name:________________

Last name: KEY

Student ID:________________

Signature:________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 100 minutes.

Q1	Q2	Q3	Q4	Total

Q1.

(3) **(a)** Find a system $\dot{x} = Ax$ and $y = Cx$ such that $y(t) = te^t$ for some $x(0)$. Specify $x(0)$.

(3) **(b)** Given

$$e^{At} = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{bmatrix}$$

find A and $(sI - A)^{-1}$.

(7) **(c)** Determine the stability of the below discrete-time systems (exponentially stable, stable, or unstable)

$$x^+ = \begin{bmatrix} e^{-1} & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x, \qquad x^+ = \begin{bmatrix} e^{-1} & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} x.$$

Answer:

a) te^t => there is a Jordan block for $\lambda = 1$ of size 2×2 or larger.

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ => $e^{At} = e^t \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

=> $y(t) = Ce^{At}x(0)$

$= [c_1 \ c_2] \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = te^t$

choose $c_1 = 1$, $c_2 = 0$, $x_1(0) = 0$, $x_2(0) = 1$

Hence, $\boxed{(A, C, x(0)) = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, [1 \ 0], \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)}$

will do the job.

b) $\left.\frac{d}{dt} e^{At}\right|_{t=0} = \left.\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e^t \end{bmatrix}\right|_{t=0} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\boxed{=> A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$

$\mathcal{L}^{-1}\{(sI - A)^{-1}\} = e^{At}$

=> $(sI - A)^{-1} = \mathcal{L}\{e^{At}\} = \boxed{\begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & 0 \\ 0 & \frac{1}{s} & 0 \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix}}$

c) First system:

$\lambda_1 = e^{-1}$ => $|\lambda_1| < 1$

$\lambda_2 = -1$ => $|\lambda_2| = 1$ size of Jordan block 1×1

$\lambda_3 = 1$ => $|\lambda_3| = 1$ " " " " 1×1

=> $\boxed{\text{stable, (NOT exp. stable)}}$

Second system:

$\lambda_1 = e^{-1}$ => $|\lambda_1| < 1$

$\lambda_2 = -1$ => $|\lambda_2| = 1$ size of Jordan block 2×2

=> $\boxed{\text{unstable}}$

Q2.

Consider the system $\dot{x} = Ax$ with $x \in \mathbb{R}^n$. For each of the below cases determine the stability of the system. *Justify your answer.*

(10) **(a)** $A^T + A < 0$.

(8) **(b)** $A^T + A = 0$.

(7) **(c)** $A^T + A > 0$.

Answer:

a) $A^T + A < 0 \implies A^T I + I A < 0$

$\implies \boxed{\text{exp. stable}}$ by Lyapunov test

b) Let $V(x) = \|x\|^2 = x^T x$

$$\implies \dot{V} = \dot{x}^T x + x^T \dot{x}$$
$$= x^T A^T x + x^T A x$$
$$= x^T (A^T + A) x$$
$$= 0$$

$\implies \|x(t)\| = \|x(0)\| \quad \forall t$

$\implies \boxed{\text{Stable but NOT exp. Stable}}$

c) $A^T + A > 0$ let $\tilde{A} = -A$

$\implies -A^T - A < 0 \implies \tilde{A}^T + \tilde{A} < 0$

by part (a) system $\dot{z} = \tilde{A} z$ is exp. Stable.

$\implies$ all the eigenvalues of $\tilde{A}$ satisfy $Re\{\tilde{\lambda}\} < 0$

$\implies$ " " " " A " $Re\{\lambda\} > 0$

$\implies \boxed{\text{unstable}}$

Q3.

Consider the homogeneous LTV system

$$\dot{x} = A(t)x\,, \qquad x(0) = x_0$$

whose state transition matrix is $\Phi(t, \tau)$. Consider also the non-homogeneous system

$$\dot{z} = A(t)z + x(t)\,, \qquad z(0) = z_0$$

whose input $x(t)$ is the solution of the homogeneous system.

(8) **(a)** Write $x(t)$ and $z(t)$ in terms of x_0, z_0, and Φ. *Your answer should not have any integral terms.*

(7) **(b)** Suppose $z(T) = 0$ for some particular $T > 0$. Show that x_0 and z_0 then must be linearly dependent.

(5) **(c)** For the augmented state $\eta := \begin{bmatrix} x \\ z \end{bmatrix}$ find the matrix $A_{\text{big}}(t)$ such that $\dot{\eta} = A_{\text{big}}(t)\eta$.

(5) **(d)** Find the state transition matrix for the system $\dot{\eta} = A_{\text{big}}(t)\eta$.

Answer:

a) $\boxed{x(t) = \Phi(t,0)\,x_0}$

$$z(t) = \Phi(t,0)\,z_0 + \int_0^t \Phi(t,\tau)x(\tau)\,d\tau$$

$$= \Phi(t,0)\,z_0 + \int_0^t \Phi(t,\tau)\Phi(\tau,0)\,x_0\,d\tau$$

$$= \Phi(t,0)\,z_0 + \int_0^t \Phi(t,0)\,x_0\,d\tau$$

$$= \Phi(t,0)\,z_0 + \Phi(t,0)\,x_0 \int_0^t d\tau$$

$$= \boxed{\Phi(t,0)\,z_0 + t\,\Phi(t,0)\,x_0}$$

b) $0 = z(T)$

$$= \Phi(T,0)\,z_0 + T\,\Phi(T,0)\,x_0$$

$$= \underbrace{\Phi(T,0)}_{\text{always nonsingular}}\{z_0 + T x_0\}$$

$$\Rightarrow \boxed{z_0 + T x_0 = 0}$$

c)

$$\dot{\eta} = \begin{bmatrix} A(t) & 0 \\ I & A(t) \end{bmatrix}\eta$$

$$\Rightarrow \boxed{A_{\text{big}}(t) = \begin{bmatrix} A(t) & 0 \\ I & A(t) \end{bmatrix}}$$

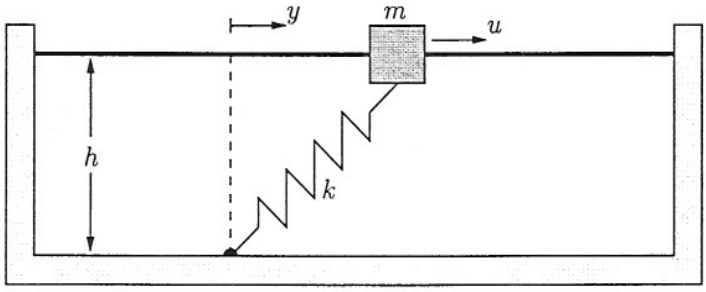
d)

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \Phi(t,0) & 0 \\ t\,\Phi(t,0) & \Phi(t,0) \end{bmatrix}\begin{bmatrix} x_0 \\ z_0 \end{bmatrix}$$

$$\Rightarrow \boxed{\Phi_{\text{big}}(t,\tau) = \begin{bmatrix} \Phi(t,\tau) & 0 \\ (t-\tau)\,\Phi(t,\tau) & \Phi(t,\tau) \end{bmatrix}}$$

Q4.

A mass m slides over a rigid bar with friction coefficient b and is connected to a spring with spring constant k. The spring is in an unstretched position when it is at an angle of 45 degrees from the vertical. The horizontal displacement of the mass from the vertical is y. The mass is subjected to a force input u.



The normalized model of this system for $m = k = h = b = 1$ is

$$\ddot{y} + \dot{y} + \left(1 - \sqrt{\frac{2}{1+y^2}}\right) y = u\,.$$

(6) **(a)** For the state choice $x_1 = y$ and $x_2 = \dot{y}$ write the state space representation of this system.

(7) **(b)** For $u = 0$, find all the equilibrium points of this system.

(7) **(c)** Obtain the linearization of the system at the origin $(y, \dot{y}) = (0, 0)$.

(5) **(d)** Determine the (local) stability of the origin.

Answer:

a) $\dot{x}_1 = x_2$

$$\dot{x}_2 = -\left(1 - \left(\frac{2}{1+x_1^2}\right)^{1/2}\right) x_1 - x_2 + u$$

$$\Rightarrow \dot{x} = \begin{bmatrix} x_2 \\ x_1\left(\sqrt{\frac{2}{1+x_1^2}} - 1\right) - x_2 + u \end{bmatrix} =: f(x,u)$$

$$y = x_1 =: g(x,u)$$

b) $f(x,0) = 0$

$$\Rightarrow \begin{bmatrix} x_2 \\ x_1\left(\sqrt{\frac{2}{1+x_1^2}} - 1\right) - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_2 = 0\,, \quad x_1 = \{0, \mp 1\}$$

$$\Rightarrow \bar{x} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$$

c) $$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ \frac{\partial}{\partial x_1}\left\{x_1\left(\sqrt{\frac{2}{1+x_1^2}} - 1\right)\right\} & -1 \end{bmatrix}$$

$$\frac{\partial}{\partial x_1}\left\{x_1\left(\sqrt{\frac{2}{1+x_1^2}} - 1\right)\right\}\Bigg|_{x=0} = \sqrt{\frac{2}{1+x_1^2}} - 1 + x_1 \cdot \frac{\partial}{\partial x_1}\sqrt{\frac{2}{1+x_1^2}}\Bigg|_{x=0}$$

$$= \sqrt{2} - 1$$

$$\Rightarrow \frac{\partial f}{\partial x}\Bigg|_{x=0} = \begin{bmatrix} 0 & 1 \\ \sqrt{2}-1 & -1 \end{bmatrix} = A$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = B\,, \quad \frac{\partial g}{\partial x} = \begin{bmatrix} 1 & 0 \end{bmatrix} = C$$

$$\frac{\partial g}{\partial u} = 0 = D$$

$\Rightarrow$ linearization:

$$\dot{\xi} = \begin{bmatrix} 0 & 1 \\ \sqrt{2}-1 & -1 \end{bmatrix}\xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\xi$$

d) $\det(sI - A) = s(s+1) - \sqrt{2} + 1 = s^2 + s - \sqrt{2} + 1$

$\lambda_1 \cdot \lambda_2 = -\sqrt{2} + 1 < 0 \Rightarrow$ one eigenvalue positive

$\Rightarrow \boxed{\text{unstable}}$