EE502 Final Exam 25 June 2022

First name:______________________________

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Read before you start:

- There are four questions.
- The duration of the examination is 120 minutes.
- The examination is closed-book.
- Please explain all your answers.

Q1	Q2	Q3	Q4	Total

Q1.

Consider the following single-input single-output system

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}}_{A} x + Bu$$
$$y = Cx.$$

(6) **(a)** Is this system internally stable?

(7) **(b)** Find a $B \in \mathbb{R}^{2\times1}$ such that the system is uncontrollable but stabilizable.

(7) **(c)** Find a nonzero $C \in \mathbb{R}^{1\times2}$ such that the system is undetectable.

(5) **(d)** Let the B matrix be what you found in part (b). Find (if possible) a nonzero initial condition $x(0) \in \mathbb{R}^2$ for which there exists a control input $u(t)$ that achieves $x(1) = 0$.

a) $|sI - A| = \begin{vmatrix} s+1 & -1 \\ -2 & s \end{vmatrix} = s(s+1) - 2 = (s-1)(s+2)$

$\Rightarrow \lambda_1 = 1$ & $\lambda_2 = -2$

$\lambda_1 > 0 \Rightarrow$ the system is $\boxed{\text{unstable.}}$

b) Find eigenvector w_2 of A^T for $\lambda_2 = -2$.

$[sI - A^T]_{s=-2} = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix}$

$\Rightarrow w_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ & $B^T w_2 = 0$

$\Rightarrow$ choose, for instance, $\boxed{B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

c) Find eigenvector v_1 of A for $\lambda_1 = 1$

$[sI - A]_{s=1} = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$

$\Rightarrow v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ & $Cv_1 = 0$

$\Rightarrow$ choose, for instance, $\boxed{C = [2 \;\; -1]}$

d) Cont. subspace S_c = range $[B \;\; AB]$

$= \text{range} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

We need $x(0) \in S_c$. Hence, choose, for instance,

$\boxed{x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

Q2.

Consider the system

$$
\begin{aligned}
\dot{x} &= \begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix}^{\rightsquigarrow A} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\rightsquigarrow B} u \\
y &= \underbrace{[\,1 \quad 0\,]}_{\rightsquigarrow C}\, x\,.
\end{aligned}
$$

(6) **(a)** Is this system a minimal realization of its transfer function?

(5) **(b)** Let $x(0) = 0$. Find (if possible) a bounded input signal $u(t)$ that results in an unbounded output $y(t)$.

(7) **(c)** Find the feedback gain $K \in \mathbb{R}^{1\times 2}$ such that the state feedback $u = -Kx$ places the closed-loop system poles at $\lambda_{1,2} = -2 \pm j2$.

(7) **(d)** Design an observer for this system (i.e., compute the observer gain $L \in \mathbb{R}^{2\times 1}$) such that the eigenvalues of the error dynamics are at $\lambda_{1,2} = -1, -2$. Write also the observer dynamics.

a) $\operatorname{rank}[B \;\; AB] = \operatorname{rank}\begin{bmatrix} 0 & 4 \\ 1 & -2 \end{bmatrix} = 2$

$\operatorname{rank}\begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} = 2$

$\Rightarrow$ The system is both controllable and observable. Hence, $\boxed{\text{YES.}}$

b) $|sI - A| = \begin{vmatrix} s-2 & -4 \\ 2 & s+2 \end{vmatrix} = s^2 + 4$

$\Rightarrow \lambda_{1,2} = \mp j2$

$\Rightarrow$ Choose, for instance, $\boxed{u(t) = \cos 2t}$

c) Desired char. poly. $d(s) = (s+2)^2 + 2^2 = s^2 + 4s + 8$

Let $K = [k_1 \;\; k_2]$

$\Rightarrow A - BK = \begin{bmatrix} 2 & 4 \\ -2-k_1 & -2-k_2 \end{bmatrix}$

$|sI - (A-BK)| = \begin{vmatrix} s-2 & -4 \\ k_1+2 & s+2+k_2 \end{vmatrix}$

$= s^2 + k_2 s + (4k_1 - 2k_2 + 4)$

$= s^2 + 4s + 8$

$\Rightarrow k_2 = 4$ & $4k_1 - 2k_2 + 4 = 8 \Rightarrow k_1 = 3$

$\Rightarrow \boxed{K = [3 \;\; 4]}$

d) Desired char. poly. $d(s) = (s+1)(s+2) = s^2 + 3s + 2$

Let $L = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}$

$\Rightarrow A - LC = \begin{bmatrix} 2-\ell_1 & 4 \\ -2-\ell_2 & -2 \end{bmatrix}$

$|sI - (A-LC)| = \begin{vmatrix} s+\ell_1-2 & -4 \\ \ell_2+2 & s+2 \end{vmatrix}$

$= s^2 + \ell_1 s + 2\ell_1 + 4\ell_2 + 4$

$= s^2 + 3s + 2$

$\Rightarrow \ell_1 = 3$ & $\ell_2 = -2 \Rightarrow \boxed{L = \begin{bmatrix} 3 \\ -2 \end{bmatrix}}$

obs. dynamics read:

$$\boxed{\dot{\hat{x}} = A\hat{x} + L(y - C\hat{x}) + Bu}$$

Q3.

Let

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

(7) **(a)** Write the Jordan form of A. *Hint:* $A^2 = ?$

(7) **(b)** Compute e^{At}.

(5) **(c)** Determine the stability of the system $x^+ = Ax$.

(6) **(d)** Find (if possible) a matrix $C \in \mathbb{R}^{1\times 3}$ such that the pair (C, A) is observable.

a) $A^2 = A \Rightarrow A(A - I) = 0$

$\Rightarrow$ minimal poly. $m(s) = s(s-1)$

rank $A = 1 \Rightarrow$ char. poly. $d(s) = s^2(s-1)$

$$\Rightarrow \boxed{J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

b) $e^{At} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \dots$ $\quad A^k = A$

$= I + A(t + \frac{t^2}{2} + \frac{t^3}{3!} + \dots)$

$\boxed{= I + (e^t - 1)A}$

c) All the eigenvalues satisfy $|\lambda| \leq 1$ and all Jordan blocks are 1x1. And there is a eigenvalue on the unit circle ($|\lambda| = 1$).

Hence the system is:

$\boxed{\text{stable but not exp. stable.}}$

d) Since $A^2 = A$ we have

$$\text{rank}\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \text{rank}\begin{bmatrix} C \\ CA \\ CA \end{bmatrix} \leq 2.$$

$\Rightarrow$ The rank of obs. matrix is never $n = 3$. Hence, $\boxed{\text{impossible.}}$

Q4.

Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Suppose $A = A^T > 0$ and the pair (A, B) is controllable. Consider now the system

$$\ddot{x} + BB^T \dot{x} + Ax = Bu\,.$$

(15) **(a)** Show that this system is controllable.

(10) **(b)** Show that this system is asymptotically stable. *Hint: Try* $P = \frac{1}{2}\begin{bmatrix} A^{-1} & 0 \\ 0 & I \end{bmatrix}$.

State space representation ?

Let $\eta := \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$. Then

$$\dot{\eta} = \underbrace{\begin{bmatrix} 0 & I \\ -A & -BB^T \end{bmatrix}}_{\tilde{A}} \eta + \underbrace{\begin{bmatrix} 0 \\ B \end{bmatrix}}_{\tilde{B}} u$$

a) Suppose the pair $(\tilde{A}, \tilde{B})$ is not controllable.

Then there exists an eigenvector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ of $\tilde{A}^T$ such that $\tilde{B}^T v = 0$.

Now, $\tilde{B}^T v = 0 \Rightarrow \begin{bmatrix} 0 & B^T \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$\Rightarrow B^T v_2 = 0 \quad (1)$

Also, $\tilde{A}^T v = \lambda v$ implies

$$\begin{bmatrix} 0 & -A^T \\ I & -BB^T \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix}$$

$\Rightarrow -A^T v_2 = \lambda v_1 \quad (*)$

& $v_1 - \underbrace{BB^T v_2}_{=0} = \lambda v_2 \Rightarrow v_1 = \lambda v_2 \quad (**)$

$(*)$ & $(**)$ $\Rightarrow A^T v_2 = -\lambda^2 v_2 \quad (2)$

(that is, v_2 is an eigenvector of A^T)

(1) & (2) contradicts the fact that the pair (A, B) is controllable. ∎

b) We can write

$$\tilde{A}P + P\tilde{A}^T = \begin{bmatrix} 0 & I \\ -A & -BB^T \end{bmatrix}\begin{bmatrix} A^{-1}/2 & 0 \\ 0 & I/2 \end{bmatrix} + \begin{bmatrix} A^{-1}/2 & 0 \\ 0 & I/2 \end{bmatrix}\begin{bmatrix} 0 & -A \\ I & -BB^T \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I/2 \\ -I/2 & -BB^T/2 \end{bmatrix} + \begin{bmatrix} 0 & -I/2 \\ I/2 & -BB^T/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & -BB^T \end{bmatrix} = -\tilde{B}\tilde{B}^T$$

Hence, $\tilde{A}P + P\tilde{A}^T = -\tilde{B}\tilde{B}^T. \quad (3)$

Since $(\tilde{A}, \tilde{B})$ is controllable, (3) implies (by Lyap. Thm) that the system is exp. stable. ∎