First name:____________________________

Last name:____________________________

Student ID:____________________________

Signature:____________________________

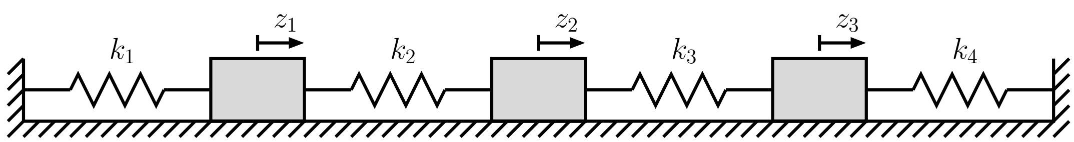
Read before you start:

- There are FIVE QUESTIONS.
- The examination is OPEN-BOOK.
- Besides correctness, the CLARITY of your presentation will also be graded.
- NO COLLABORATION with others.

Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Total**

Q1.

Consider the LTI mass spring system below, where there is no friction. The blocks are identical with unit mass, i.e., $m_1 = m_2 = m_3 = 1$. The displacements of the blocks from the equilibrium positions are denoted by $z_1,\ z_2,\ z_3 \in \mathbb{R}$ as shown in the figure. The spring constants are denoted by $k_i > 0$ for $i = 1, 2, 3, 4$. For each of the following cases find (if exists) a condition on the spring constants so that the system is unobservable from the indicated output y.



(a) $y = z_1$.

(b) $y = z_2$.

Sol'n. The dynamics of the system read

$$\ddot{z}_1 + k_1 z_1 + k_2(z_1 - z_2) = 0 \tag{1}$$
$$\ddot{z}_2 + k_2(z_2 - z_1) + k_3(z_2 - z_3) = 0 \tag{2}$$
$$\ddot{z}_3 + k_3(z_3 - z_2) + k_4 z_3 = 0\,. \tag{3}$$

Recall that a system is unobservable if and only if we can find a nonzero solution which produces an identically zero output $y(t) \equiv 0$; see HW4 Q1(a). This allows us to proceed as follows.

(a) Let $y \equiv 0$. Then $z_1 \equiv 0 \implies \dot{z}_1 \equiv 0 \implies \ddot{z}_1 \equiv 0$. By (1) this implies $-k_2 z_2 \equiv 0$. Since $k_2 > 0$ we can write $z_2 \equiv 0$ yielding $\ddot{z}_2 \equiv 0$. Now, under $z_1 \equiv 0$, $z_2 \equiv 0$, and $\ddot{z}_2 \equiv 0$ we have $-k_3 z_3 \equiv 0$ thanks to (2). Then we can write $z_3 \equiv 0$ because k_3 is nonzero. To sum up, we have discovered that $y \equiv 0$ implies $z_1,\ z_2,\ z_3 \equiv 0$ (and, consequently, $\dot{z}_1,\ \dot{z}_2,\ \dot{z}_3 \equiv 0$). In other words, only the system at rest produces $y \equiv 0$. Hence the system is observable from $y = z_1$ regardless of the spring constants.

(b) Let $y \equiv 0$. Then we have $z_2,\ \ddot{z}_2 \equiv 0$. Rewriting the dynamics under this condition we have

$$\ddot{z}_1 + (k_1 + k_2)z_1 = 0$$
$$k_2 z_1 + k_3 z_3 = 0$$
$$\ddot{z}_3 + (k_3 + k_4)z_3 = 0\,.$$

Can these equations be satisfied by some nonzero solution? Yes. Let $k_1 + k_2 = k_3 + k_4 = \omega^2$. Then $z_1(t) = k_3 \cos(\omega t)$ and $z_3(t) = -k_2 \cos(\omega t)$ satisfy the above equations. Hence the system is unobservable from $y = z_2$ if the spring constants satisfy $k_1 + k_2 = k_3 + k_4$.

Q2.

Let P, Q, $R \in \mathbb{R}^{n\times n}$ be symmetric positive definite matrices.

(a) Find (if possible) a pair (P, Q) such that the following system is unstable. Explain if impossible.

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & -P \\ Q & 0 \end{bmatrix}}_{A_1} x$$

(b) Find (if possible) a triple (P, Q, R) such that the following system is stable. Explain if impossible.

$$\dot{x} = \underbrace{\begin{bmatrix} P & -R \\ R & Q \end{bmatrix}}_{A_2} x$$

Sol'n. **(a)** Let the matrix $G \in \mathbb{R}^{2n\times 2n}$ be

$$G = \begin{bmatrix} Q & 0 \\ 0 & P \end{bmatrix}.$$

Note that G is symmetric positive definite and satisfies $A_1^T G + GA_1 = 0$. Let $x(t)$ be an arbitrary solution of the system $\dot{x} = A_1 x$. Construct the scalar function $V(x(t)) = x(t)^T Gx(t)$. We can write $\dot{V} = \dot{x}^T Gx + x^T G\dot{x} = x^T(A_1^T G + GA_1)x = 0$. Therefore $V(x(t)) = V(x(0))$ for all $t \geq 0$. This allows us to write $\|x(t)\|^2 \leq \lambda_{\min}(G)^{-1}V(x(t)) = \lambda_{\min}(G)^{-1}V(x(0)) \leq \lambda_{\min}(G)^{-1}\lambda_{\max}(G)\|x(0)\|^2$. That is, $\|x(t)\| \leq \sqrt{\lambda_{\min}(G)^{-1}\lambda_{\max}(G)}\|x(0)\|$ for all $t \geq 0$. Since all its solutions are bounded the system $\dot{x} = A_1 x$ is stable regardless of the choice (P, Q).

(b) Let $H = -A_2$. We then have

$$H^T + H = -2\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}.$$

Therefore we can write $H^T I + IH < 0$, meaning the Lyapunov inequality is satisfied. In other words, the system $\dot{z} = Hz$ is exponentially stable. As a result, all the eigenvalues $\lambda_i(H)$ are on the open left half-plane. Then, since $\lambda_i(A_2) = -\lambda_i(H)$, all the eigenvalues $\lambda_i(A_2)$ have positive real parts. The system $\dot{x} = A_2 x$ therefore is unstable for all (P, Q, R).

Q3.

Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Consider the discrete-time system $x(k+1) = Ax(k) + Bu(k)$. Suppose this system is stabilizable. Hence we can find a feedback gain $K_1 \in \mathbb{R}^{m\times n}$ such that under the feedback law $u(k) = -K_1 x(k)$ the closed-loop system is exponentially stable. Does this imply that we can find a feedback gain $K_2 \in \mathbb{R}^{m\times n}$ such that the closed-loop system under $u(k) = -K_2 x(k-1)$ is exponentially stable? Explain.

Sol'n. No. Here is a counterexample. Consider the first order system $x(k+1) = 4x(k) + 1u(k)$ where $A = 4$ and $B = 1$. The system clearly is controllable (hence stabilizable). Under $u(k) = -K_2 x(k-1)$ we obtain the closed-loop system $x(k+1) = 4x(k) - K_2 x(k-1)$. Letting $\eta(k) = [x(k-1) \;\; x(k)]^T$ yields the state space representation

$$\eta^+ = \underbrace{\begin{bmatrix} 0 & 1 \\ -K_2 & 4 \end{bmatrix}}_{A_{\rm cl}} \eta\,.$$

For exponential stability both eigenvalues λ_1 and λ_2 of $A_{\rm cl}$ should be placed inside the unit circle by a proper choice of K_2. Note that the characteristic polynomial of $A_{\rm cl}$ reads $d(s) = s^2 - 4s + K_2 = s^2 - (\lambda_1 + \lambda_2)s + \lambda_1\lambda_2$, which tells us that $\lambda_1 + \lambda_2 = 4$ regardless of K_2. Now, clearly, $\lambda_1 + \lambda_2 = 4$ means at least one eigenvalue will always be outside the unit circle no matter how we choose the gain K_2. Hence this system cannot be stabilized by delayed state feedback.

Q4.

Consider the following system

$$\begin{aligned} \dot{x} &= \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 2 \\ 1 & 1 & -2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u \\ y &= \underbrace{[1 \ \ -1 \ \ 0]}_{C}\, x\,. \end{aligned}$$

(a) Is this system detectable? Explain.

(b) Is this system BIBO stable? Explain.

Sol'n. **(a)** No. Note that $v = [1 \ \ 1 \ \ 1]^T$ is the eigenvector for the eigenevalue at the origin, i.e., $Av = 0$. Moreover, $Cv = 0$. That is, A has an eigenvector in null C whose eigenvalue satisfies $\mathrm{Re}(\lambda) \geq 0$. By the eigenvector test therefore the system is undetectable.

(b) Yes. The transfer function of this system reads

$$H(s) = C(sI - A)^{-1}B = \frac{-2}{(s+1)(s+2)}\,.$$

Since both poles $\lambda_1 = -1$ and $\lambda_2 = -2$ satisfy $\mathrm{Re}(\lambda_i) < 0$, the system is BIBO stable.

Q5.

Consider the following system

$$\begin{aligned} x^+ &= \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 2 \\ 1 & 1 & -2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u \\ y &= \underbrace{[1 \;\; -1 \;\; 0]}_{C}\, x \,. \end{aligned}$$

Design the observer gain $L \in \mathbb{R}^{3\times 1}$ and the feedback gain $K \in \mathbb{R}^{1\times 3}$ such that under the observer-based feedback law $u = -K\hat{x}$ (where $\hat{x}$ is the state estimate generated by your observer) the closed-loop system is finite-time stable. That is, there exists some finite integer N such that all the solutions satisfy $x(k) = 0$ for $k \geq N$ regardless of the initial conditions $x(0)$ and $\hat{x}(0)$. Write also your observer dynamics.

Sol'n. The observer dynamics read $\hat{x}^+ = A\hat{x} - BK\hat{x} + L(y - C\hat{x})$. Hence the overall closed-loop dynamics can be written as

$$\begin{bmatrix} x \\ \hat{x} \end{bmatrix}^+ = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} .$$

Rewriting the above equation in terms of x and the error $e = \hat{x} - x$ produces

$$\begin{bmatrix} x \\ e \end{bmatrix}^+ = \underbrace{\begin{bmatrix} A - BK & -BK \\ 0 & A - LC \end{bmatrix}}_{\Phi} \begin{bmatrix} x \\ e \end{bmatrix} .$$

Finite-time stability can be achieved by making Φ nilpotent. Since Φ is block triangular, this is possible by making the individual blocks $A - BK$ and $A - LC$ on the diagonal nilpotent. In other words, the gains K and L should be chosen such that all the eigenvalues of $A - BK$ and $A - LC$ are at the origin. Below then is the answer.

$$K = [2.5 \;\; 0.5 \;\; -3] \,, \qquad L = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where $\alpha \in \mathbb{R}$ is arbitrary.