First name:__________

Last name: KEY

Student ID:__________

Signature:__________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 120 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS.

Q1	Q2	Q3	Q4	Total

Q1.

Consider the system below. [Observe that $A^2 = 0$.]

$$\dot{x} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -3 & -3 & -3 \end{bmatrix}}_{A} x$$

$$y = \underbrace{[1 \;\; 0 \;\; -1]}_{C}\, x.$$

a) Find a basis for null A and a basis for range A.

b) Write the Jordan form of A.

c) Compute the solution $x(t)$ and the output $y(t)$ for the initial condition $x(0) = [1 \;\; 1 \;\; 1]^T$.

d) Find an initial condition $x(0) \neq [1 \;\; 1 \;\; 1]^T$ for which the output $y(t)$ is identical to the output that you computed in part (c).

a) $\text{null } A = \text{null } [1 \;\; 1 \;\; 1] = \boxed{\text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}}$

$\text{range } A = \boxed{\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right\}}$

b) $A^2 = 0$ & $A \neq 0 \Rightarrow \boxed{J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$

c) $e^{At} = I + At + \underbrace{\frac{A^2t^2}{2} + \cdots}_{=0} = I + At$

$\Rightarrow x(t) = (I + tA)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix}}$

& $y(t) = [1 \;\; 0 \;\; -1]\, x(t) = \boxed{12t}$

d) $W = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 4 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

unobs. subspace $= \text{null } W = \text{null} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$= \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow$ choose $\boxed{x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$ for any $\alpha \neq 0$.

Q2.

Consider the system below.

$$\dot{x} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = [0\ 1]x.$$

a) Compute the state transition matrix.

b) Compute the impulse response.

c) Find (if exists) a bounded input $u(t)$ for which the solution $x(t)$ is unbounded (regardless of the initial condition).

d) Find (if exists) the range of the gain $k \in \mathbb{R}$ for which the output feedback $u = -ky$ makes the closed-loop system exponentially stable (in the sense of Lyapunov).

a) $sI - A = \begin{bmatrix} s-1 & 2 \\ -1 & s+1 \end{bmatrix}$

$$[sI-A]^{-1} = \frac{1}{s^2+1}\begin{bmatrix} s+1 & -2 \\ 1 & s-1 \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}\{[sI-A]^{-1}\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{s+1}{s^2+1} & \frac{-2}{s^2+1} \\ \frac{1}{s^2+1} & \frac{s-1}{s^2+1} \end{bmatrix}\right\}$$

$$= \boxed{\begin{bmatrix} \cos t + \sin t & -2\sin t \\ \sin t & \cos t - \sin t \end{bmatrix}}$$

b) $h(t) = Ce^{At}B = [0\ 1]e^{At}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \boxed{\cos t - \sin t}$

c) Choose, for instance, $\boxed{u(t) = \cos t}$

d) Closed-loop system matrix $A_{cl} = A - kBC$

$\Rightarrow A_{cl} = \begin{bmatrix} 1 & -2 \\ 1 & -1-k \end{bmatrix}$

$\Rightarrow |sI - A_{cl}| = \begin{vmatrix} s-1 & 2 \\ -1 & s+1+k \end{vmatrix} = (s-1)(s+1+k) + 2$

$= s^2 + ks + 1 - k = s^2 + a_1 s + a_2$

We need $a_1 > 0$ & $a_2 > 0$

Hence $\boxed{0 < k < 1}$

Q3.

Consider the system below.

$$x^+ = \underbrace{\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \underbrace{[0 \;\; 1]}_{C} x.$$

a) Can you find a symmetric positive definite matrix P satisfying $A^T PA - P = -C^T C$?

b) Can you find a symmetric positive definite matrix P satisfying $\frac{1}{4}APA^T - P = -BB^T$?

c) Let $x(0) = [-8 \;\; 3]^T$. Compute the input sequence $(u(0), u(1), \ldots)$ such that $x(k) = 0$ for $k = 2, 3, \ldots$

d) For this system construct an observer (whose state we denote by $\hat{x}$) such that for all initial conditions $x(0)$, $\hat{x}(0)$ and all input sequences $(u(0), u(1), \ldots)$ we have $\hat{x}(k) = x(k)$ for $k = 2, 3, \ldots$

a) $W = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. rank $W = 2 = n$ $\Rightarrow$ sys. observable

$|sI - A| = \begin{vmatrix} s-1 & 2 \\ -1 & s+1 \end{vmatrix} = s^2+1 \Rightarrow \lambda_{1,2} = \mp j$

$|\lambda_i| \geq 1 \Rightarrow$ sys. NOT exp. stable.

Hence, $\boxed{\text{NO}}$

b) Let $\tilde{A} = \frac{1}{2}A$. Then

$\frac{1}{4}APA^T - P = -BB^T \Leftrightarrow \tilde{A}P\tilde{A}^T - P = -BB^T$

$Q = [B \;\; \tilde{A}B] = \begin{bmatrix} 0 & -1 \\ 1 & -1/2 \end{bmatrix}$. rank $Q = 2 = n$ $\Rightarrow$ pair $(\tilde{A}, B)$ cont.

Also, we have $\tilde{\lambda}_{1,2} = \mp \frac{1}{2} j \Rightarrow |\tilde{\lambda}_i| < 1 \;\; \forall i$

$\Rightarrow \tilde{A}$ is exp. stable

Hence, $\boxed{\text{YES}}$

c) $x(1) = Ax(0) + Bu(0)$

$x(2) = Ax(1) + Bu(1) = A\{Ax(0) + Bu(0)\} + Bu(1)$

$= A^2 x(0) + \begin{bmatrix} AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \end{bmatrix}$

$x(2) = 0 \Rightarrow \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} = -[AB \;\; B]^{-1} A^2 x(0)$

$[AB \;\; B]^{-1} = \begin{bmatrix} -2 & 0 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -1/2 & 0 \\ -1/2 & 1 \end{bmatrix}$.

$A^2 = -I$

$\Rightarrow \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} = \begin{bmatrix} -1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} -8 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

$\Rightarrow (u(0), u(1), u(2), \ldots) = \boxed{(4, 7, 0, 0, \ldots)}$

d) $\boxed{\hat{x}^+ = A\hat{x} + Bu + L(y - C\hat{x})}$ obs. dynamics

$\hat{x}(k) = x(k)$ for $k \geq 2 \Rightarrow (A - LC)^2 = 0$

$\Rightarrow |sI - [A - LC]| = s^2$

Let $L = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \Rightarrow A - LC = \begin{bmatrix} 1 & -2-\ell_1 \\ 1 & -1-\ell_2 \end{bmatrix}$

$|sI - [A - LC]| = \begin{vmatrix} s-1 & 2+\ell_1 \\ -1 & s+1+\ell_2 \end{vmatrix} = (s-1)(s+1+\ell_2) + 2 + \ell_1$

$= s^2 + \ell_2 s - 1 - \ell_2 + 2 + \ell_1 = s^2$

$\Rightarrow \ell_2 = 0$ & $\ell_1 = -1$

$\Rightarrow \boxed{L = \begin{bmatrix} -1 \\ 0 \end{bmatrix}}$

Q4.

Consider the third-order single-input single-output LTI system below. ["*" means "don't care."]

$$\dot{x} = \begin{bmatrix} 3 & * & * \\ -1 & * & * \\ -3 & 3 & -5 \end{bmatrix} x + Bu$$
$$y = Cx\,.$$

The following are given:

- The unobservable subspace is $\mathcal{S}_{\bar{o}} = \mathrm{range}\,[2 \;\; 0 \; -1]^T$.
- The controllable subspace is $\mathcal{S}_c = \mathrm{null}\,[-1 \;\; 2 \; -2]$.
- The characteristic polynomial is $d(s) = (s - \lambda_1)(s - \lambda_2)(s+2)$.

Answer the questions below.

a) Is this system detectable?

b) Is this system stabilizable?

c) Is this system stable in the sense of Lyapunov?

d) Is this system BIBO stable?

a) Sys. unobs. $\Rightarrow \exists\, v_1 \neq 0 \quad Av_1 = \lambda_1 v_1 \;\&\; Cv_1 = 0$

$$\Rightarrow \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}}_{W} v_1 = 0 \Rightarrow v_1 \in \mathrm{null}\, W = S_{\bar{o}}$$

$$\Rightarrow v_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad \lambda_1 = ?$$

$$\begin{bmatrix} 3 & * & * \\ -1 & * & * \\ -3 & 3 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} * \\ * \\ -1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \lambda_1 = 1$$

Since $\lambda_1 > 0$, sys. $\boxed{\text{NOT detectable}}$

b) Sys. uncont. $\Rightarrow \exists\, w_2 \neq 0 \quad A^T w_2 = \lambda_2 w_2 \;\&\; B^T w_2 = 0$

By dual arguments we have $w_2 = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, \; \lambda_2 = ?$

$$\begin{bmatrix} 3 & -1 & -3 \\ * & * & 3 \\ * & * & -5 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ * \\ * \end{bmatrix} = \lambda_2 \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \Rightarrow \lambda_2 = -1$$

Since $\lambda_2 < 0$, sys. $\boxed{\text{stabilizable}}$

c) $\lambda_1 > 0 \Rightarrow$ sys. $\boxed{\text{unstable}}$

d) λ_1 unobs. & λ_2 uncont.

$\Rightarrow H(s) = \dfrac{*}{s+2} \Rightarrow$ the only pole $= -2 < 0$

$\Rightarrow$ sys. $\boxed{\text{BIBO stable}}$