First name:____________________________

Last name:____________________________

Student ID:____________________________

Signature:____________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 120 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

Consider the system below. [Observe that $A^2 = 0$.]

$$\begin{aligned}\dot{x} &= \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -3 & -3 & -3 \end{bmatrix}}_{A} x \\ y &= \underbrace{[1 \;\; 0 \;\; -1]}_{C}\, x\,.\end{aligned}$$

a) Find a basis for null A and a basis for range A.

b) Write the Jordan form of A.

c) Compute the solution $x(t)$ and the output $y(t)$ for the initial condition $x(0) = [1 \;\; 1 \;\; 1]^T$.

d) Find an initial condition $x(0) \neq [1 \;\; 1 \;\; 1]^T$ for which the output $y(t)$ is identical to the output that you computed in part (c).

Q2.

Consider the system below.

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= [0 \;\; 1]\, x\,. \end{aligned}$$

a) Compute the state transition matrix.

b) Compute the impulse response.

c) Find (if exists) a bounded input $u(t)$ for which the solution $x(t)$ is unbounded (regardless of the initial condition).

d) Find (if exists) the range of the gain $k \in \mathbb{R}$ for which the output feedback $u = -ky$ makes the closed-loop system exponentially stable (in the sense of Lyapunov).

Q3.

Consider the system below.

$$\begin{aligned} x^+ &= \underbrace{\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \\ y &= \underbrace{[0 \;\; 1]}_{C}\, x\,. \end{aligned}$$

a) Can you find a symmetric positive definite matrix P satisfying $A^TPA - P = -C^TC$?

b) Can you find a symmetric positive definite matrix P satisfying $\frac{1}{4}APA^T - P = -BB^T$?

c) Let $x(0) = [\,-8 \;\; 3\,]^T$. Compute the input sequence $(u(0),\, u(1),\, \ldots)$ such that $x(k) = 0$ for $k = 2,\, 3,\, \ldots$

d) For this system construct an observer (whose state we denote by $\hat{x}$) such that for all initial conditions $x(0)$, $\hat{x}(0)$ and all input sequences $(u(0),\, u(1),\, \ldots)$ we have $\hat{x}(k) = x(k)$ for $k = 2,\, 3,\, \ldots$

Q4.

Consider the third-order single-input single-output LTI system below. ["$*$" means "don't care."]

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 3 & * & * \\ -1 & * & * \\ -3 & 3 & -5 \end{bmatrix} x + Bu \\ y &= Cx\,. \end{aligned}$$

The following are given:

- The unobservable subspace is $\mathcal{S}_{\bar{\text{o}}} = \text{range}\,[\,2 \;\; 0 \; -1\,]^T$.
- The controllable subspace is $\mathcal{S}_{\text{c}} = \text{null}\,[\,-1 \;\; 2 \; -2\,]$.
- The characteristic polynomial is $d(s) = (s - \lambda_1)(s - \lambda_2)(s + 2)\,$.

Answer the questions below.

a) Is this system detectable?

b) Is this system stabilizable?

c) Is this system stable in the sense of Lyapunov?

d) Is this system BIBO stable?