First name:

Last name: KEY

Student ID:

Signature:

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 150 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS.

Q1	Q2	Q3	Q4	Total

Q1.

Consider the system below.

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = [2 \quad 1] x.$$

a) Is this system internally stable?

b) Is this system BIBO stable?

c) Is this system observable?

d) Is this system detectable?

a) $|sI - A| = \begin{vmatrix} s+1 & -1 \\ -2 & s+2 \end{vmatrix} = (s+1)(s+2) - 2 = s(s+3)$

$\lambda_{1,2} = 0, -3 \Rightarrow$ $\boxed{\text{internally stable}}$

b) $TF = C(sI-A)^{-1}B = \dfrac{[2 \quad 1]\begin{bmatrix} s+2 & 1 \\ 2 & s+1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{s(s+3)}$

$= \dfrac{2s+6}{s(s+3)} = \dfrac{2}{s} \Rightarrow$ a pole at the origin

$\Rightarrow$ $\boxed{\text{NOT BIBO stable}}$

c) obs. matrix $W = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

$\det W = 0 \Rightarrow$ $\boxed{\text{NOT observable}}$

d) check the eigenvector of $\lambda_1 = 0$

$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} v_1 = 0 \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Cv_1 = [2 \quad 1]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \neq 0$

$v_1 \notin \text{null } C \Rightarrow$ $\boxed{\text{detectable}}$

Q2.

Consider the following system with impulse response $h(t) = 1 + e^t$.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx\,. \end{aligned}$$

(5) **a)** Find a suitable triple $(A,\, B,\, C)$ such that the system is a minimal realization of $H(s)$.

(5) **b)** For your design in part (a) find a feedback gain K such that under the state feedback $u = -Kx$ all the closed-loop eigenvalues are located at $\lambda = -1$.

(7) **c)** For your design in part (a) construct an observer such that the eigenvalues of the error dynamics are located at $\lambda = -1$.

(8) **d)** Find a suitable triple $(A,\, B,\, C)$ that is a realization of $H(s)$ and satisfies all of the following constraints:

- A has an eigenvalue at $\lambda = -2$,
- The system is uncontrollable but stabilizable,
- The unobservable subspace contains the vector $[1\ \ 0\ \ 1]^T$.

a) $H(s) = \frac{1}{s} + \frac{1}{s-1} = \frac{2s-1}{s^2-s}$

$\frac{Y}{U} = \frac{2s-1}{s^2-s} \Rightarrow$ Define $X_1 := \frac{1}{s^2-s}U$, $X_2 := sX_1$

$$\left.\begin{aligned} \Rightarrow\ sX_1 &= X_2 \\ sX_2 &= s^2X_1 = sX_1 + U = X_2 + U \end{aligned}\right\} (1)$$

Also, $Y = (2s-1)X_1 = 2sX_1 - X_1 = -X_1 + 2X_2$ (2)

(1)&(2) $\Rightarrow$
$$\boxed{\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= [-1\ \ 2]\,x \end{aligned}}$$

b) Let $K = [k_1\ \ k_2]$

$\Rightarrow A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & 1-k_2 \end{bmatrix}$

$\Rightarrow |sI - [A-BK]| = s(s+k_2-1) + k_1 = s^2 + (k_2-1)s + k_1$

desired char. poly. $d(s) = (s+1)^2 = s^2 + 2s + 1$

$\Rightarrow k_1 = 1,\ k_2 = 3 \Rightarrow \boxed{K = [1\ \ 3]}$

c) Let $L = [l_1\ \ l_2]^T$

$\Rightarrow A - LC = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -l_1 & 2l_1 \\ -l_2 & 2l_2 \end{bmatrix} = \begin{bmatrix} l_1 & 1-2l_1 \\ l_2 & 1-2l_2 \end{bmatrix}$

$\Rightarrow |sI - (A-LC)| = (s-l_1)(s+2l_2-1) + l_2(2l_1-1)$

$= s^2 + (2l_2 - l_1 - 1)s + l_1 - l_2 = s^2 + 2s + 1$

$$\Rightarrow \left.\begin{aligned} -l_1 + 2l_2 &= 3 \\ l_1 - l_2 &= 1 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} l_1 &= 5 \\ l_2 &= 4 \end{aligned}\right\} \boxed{L = \begin{bmatrix} 5 \\ 4 \end{bmatrix}}$$

observer dynamics: $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$

d) Use controllable decomposition

$A = \left[\begin{array}{cc:c} 0 & 1 & a \\ 0 & 1 & b \\ \hdashline 0 & 0 & -2 \end{array}\right]$, $B = \left[\begin{array}{c} 0 \\ 1 \\ \hdashline 0 \end{array}\right]$, $C = [-1\ \ 2 \,\vdots\, d]$

$a, b, d = ?$

$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow [-1\ \ 2\ \ d]\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0 \Rightarrow d = 1$

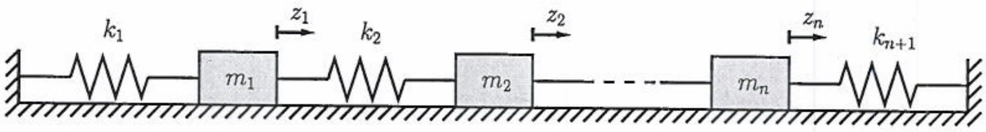
$CA\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = [0\ \ 1\ \ -a+2b-2]\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0 \Rightarrow -a + 2b = 2$

$CA^2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = [0\ \ 1\ \ b-2(-a+2b-2)]\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0 \Rightarrow 2a - 3b = -4$

$\Rightarrow b = 0,\ a = -2$

Hence,
$$\boxed{A = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ C = [-1\ \ 2\ \ 1]}$$

Q3.



Consider the coupled mass spring systems above whose dynamics read $M\ddot{z} + C^TC\dot{z} + Kz = 0$, where $z = [z_1\ z_2\ \ldots\ z_n]^T \in \mathbb{R}^n$ and the matrices M, $K \in \mathbb{R}^{n\times n}$ are symmetric positive definite. The viscous friction is represented by $C \in \mathbb{R}^{q\times n}$.

(5) **a)** For the state choice $x = [x_1^T\ x_2^T]^T$ with $x_1 = z$ and $x_2 = \dot{z}$ obtain the state space representation $\dot{x} = Ax$.

(5) **b)** Employing the matrix $P = \begin{bmatrix} K/2 & 0_{n\times n} \\ 0_{n\times n} & M/2 \end{bmatrix}$ show that this system is stable. *Note that x^TPx is the total energy of the system.*

(10) **c)** Show that if the pair $(C, M^{-1}K)$ is observable then so is the pair $([0_{q\times n}\ C], A)$.

(5) **d)** Show that if the pair $(C, M^{-1}K)$ is observable then the system is exponentially stable.

a) $\boxed{\dot{x} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C^TC \end{bmatrix} x}$

b) $A^TP + PA = \frac{1}{2}\begin{bmatrix} 0 & -KM^{-1} \\ I & -C^TCM^{-1} \end{bmatrix}\begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix}$

$+ \frac{1}{2}\begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C^TC \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 0 & -K \\ K & -C^TC \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & K \\ -K & -C^TC \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ 0 & C^TC \end{bmatrix}$

$\Rightarrow A^TP + PA \leq 0 \Rightarrow$ system stable. ▨

c) Suppose not. Then we can find a nonzero

$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{C}^{2n}$ such that:

$Av = \lambda v$ $(\lambda \in \mathbb{C})$ & $[0\ C]v = 0$

Now, $[0\ C]v = 0 \Rightarrow Cv_2 = 0$

& $Av = \lambda v \Rightarrow \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C^TC \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix}$

$\Rightarrow v_2 = \lambda v_1$ & $-M^{-1}Kv_1 - M^{-1}C^T\underbrace{Cv_2}_{=0} = \lambda v_2 = \lambda^2 v_1$

$\Rightarrow -M^{-1}Kv_1 = \lambda^2 v_1 \Rightarrow v_1$ eigenvector of $M^{-1}K$ (1)

Also, $\lambda = 0$ because $M^{-1}K$ is nonsingular.

$\Rightarrow Cv_2 = 0 \Rightarrow C\lambda v_1 = 0 \Rightarrow Cv_1 = 0$ (2)

(1) & (2) $\Rightarrow$ the pair $(C, M^{-1}K)$ is unobservable

The result follows by contradiction. ▨

d) Let $\tilde{C} = [0\ C]$.

We've obtained in part (b): $A^TP + PA = -\tilde{C}^T\tilde{C}$ (3)

We've obtained in part (c): $(\tilde{C}, A)$ observable (4)

(3) & (4) $\Rightarrow$ exp. stability ▨

Q4.

Let

$$A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$$

a) Find the minimal polynomial and eigenvector(s) of A.

b) Is the system $\dot{x} = Ax$ stable?

c) Is the system $x^+ = Ax$ stable?

d) Find (if possible) a symmetric positive definite matrix $P \in \mathbb{R}^{2\times 2}$ satisfying $A^T P A - P < 0$.

a) $|sI - A| = \begin{vmatrix} s-2 & 4 \\ -1 & s+2 \end{vmatrix} = (s-2)(s+2) + 4 = s^2$

$A \neq 0 \Rightarrow \boxed{m(s) = s^2}$

$Av = 0 \Rightarrow \boxed{v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$

b) Jordan form of A is $J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}_{2\times 2}$

$\Rightarrow \dot{x} = Ax$ is $\boxed{\text{unstable}}$

c) Eigenvalues satisfy $|\lambda_i| < 1$

$\Rightarrow x^+ = Ax$ is $\boxed{\text{exp. stable}}$

d) choose $P = I + A^T A + (A^2)^T A^2 + (A^3)^T A^3 + \dots$

$= I + A^T A$ (because $A^2 = 0$)

$\Rightarrow P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & -10 \\ -10 & 21 \end{bmatrix}}$

(Note that $A^T P A - P = -I$)