First name:____________________

Last name: KEY

Student ID:____________________

Signature:____________________

Read before you start:

- There are four questions.
- The examination is open-book.
- No computer/calculator is allowed.
- The duration of the examination is 150 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS.

Q1	Q2	Q3	Q4	Total

Q1. 25%

Consider the following system.

$$\dot{x} = \overbrace{\begin{bmatrix} \boxed{\begin{matrix} 3 & -4 \\ 2 & -3 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 1 & 0 \end{matrix} & \underbrace{\boxed{2}}_{A_u} \end{bmatrix}}^{A_o} x + \overbrace{\begin{bmatrix} \boxed{\begin{matrix} 2 \\ 1 \end{matrix}} \\ 0 \end{bmatrix}}^{B_o} u = Ax + Bu$$

$$y = [\underbrace{\boxed{0 \;\; 1}}_{C_o} \;\; 0]\, x.$$

(a) Is this system controllable?

(b) Is this system observable?

(c) Can we stabilize this system by state feedback $u = -Kx$?

(d) Can we asymptotically estimate the state of this system by an observer?

(e) Is this system BIBO stable?

a) $AB = \begin{bmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$A^2B = A(AB) = \begin{bmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$

con. matrix $= [B \;\; AB \;\; A^2B] = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 2 & 6 \end{bmatrix}$ lin. dep. rows

rank $< 3 \Rightarrow$ $\boxed{\text{uncontrollable}}$

b) system is in observable decomp. form with $A_u \in \mathbb{R}^{1\times 1}$. Since A_u exists system is $\boxed{\text{observable}}$

c) check stabilizability.

$\lambda(A) = \lambda(A_o) \cup \lambda(A_u)$

$\rightarrow = 2$

$|sI - A_o| = \begin{vmatrix} s-3 & 4 \\ -2 & s+3 \end{vmatrix} = s^2 - 9 + 8 = s^2 - 1 = (s-1)(s+1)$

$\Rightarrow \lambda(A) = \{1, -1, 2\} = \lambda(A^T)$

unstable eigenvalues: $\lambda_1 = 1$ & $\lambda_3 = 2$. Find the corresponding eigenvectors (v_1, v_3) of A^T.

$\underline{\lambda_1 = 1}$ $\quad 0 = [A^T - I]v_1 = \begin{bmatrix} 2 & 2 & 1 \\ -4 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\Rightarrow v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$\underline{\lambda_3 = 2}$ $\quad 0 = [A^T - 2I]v_3 = \begin{bmatrix} 1 & 2 & 1 \\ -4 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$

$\Rightarrow v_3 = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

Now, $B^T v_1 = [2 \;\; 1 \;\; 0] \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 1 \neq 0$

& $B^T v_3 = [2 \;\; 1 \;\; 0] \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} = 6 \neq 0$

Since $v_1 \notin \mathcal{N}(B^T)$ & $v_3 \notin \mathcal{N}(B^T)$, $\boxed{\text{stabilizable}}$

$\Rightarrow$ $\boxed{\text{YES}}$

d) $\boxed{\text{NO}}$ because system is undetectable due to $\lambda(A_u) = 2 > 0$.

e) $TF = C_o(sI - A_o)^{-1} B_o$

$= \dfrac{[0 \;\; 1] \begin{bmatrix} s+3 & -4 \\ 2 & s-3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}{s^2 - 1} = \dfrac{[2 \;\; s-3] \begin{bmatrix} 2 \\ 1 \end{bmatrix}}{(s-1)(s+1)}$

$= \dfrac{\cancel{s+1}}{(s-1)\cancel{(s+1)}} = \dfrac{1}{s-1}$

pole at $\lambda = 1 > 0 \Rightarrow$ $\boxed{\text{not BIBO stable}}$

Q2. 25 %

Consider the following system.

$$\dot{x} = \begin{bmatrix} \alpha & 8 \\ -4 & \beta \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$
$$y = [4 \;\; -1]\, x.$$

It is known that this system is observable and BIBO stable. Also, for some initial condition $x(0) = \eta$ and zero input $u = 0$, the output is measured to be $y(t) = 9e^t$.

(a) Find α and β.

(b) Find η.

$y(t) = 9e^t \Rightarrow \lambda_1 = 1$ is a eigenvalue (1)

BIBO stable $\Rightarrow \lambda_1 = 1$ is not a pole of TF

$\Rightarrow$ pole-zero cancellation

$\Rightarrow$ system uncontrollable (because it is observable)

uncontrollable $\Rightarrow$ rank $[B \;\; AB] = 1$

$\Rightarrow AB = \lambda B$ (2)

impulse response $= Ce^{At}B = C(e^{\lambda t}B) = e^{\lambda t}CB$

observability $\Rightarrow CB \neq 0$ (because B is eigenvector)

BIBO stability $\Rightarrow \lambda < 0$ (3)

— o —

a) (2) $\Rightarrow$ $\begin{bmatrix} \alpha & 8 \\ -4 & \beta \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\alpha+8 \\ \beta-8 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\Rightarrow \dfrac{2\alpha+8}{\beta-8} = 2 \Rightarrow \beta = \alpha + 12$ (4)

(3) $\Rightarrow 2\alpha + 8 < 0 \Rightarrow \alpha < -4$ (5)

(1) $\Rightarrow \det[A - I] = 0$ (6)

(4) & (6) $\Rightarrow 0 = \begin{vmatrix} \alpha-1 & 8 \\ -4 & \alpha+11 \end{vmatrix} = (\alpha-1)(\alpha+11) + 32$

$= \alpha^2 + 10\alpha + 21$

$= (\alpha+3)(\alpha+7)$ (7)

(5) & (7) $\Rightarrow \boxed{\alpha = -7} \Rightarrow \boxed{\beta = 5}$

b) $y(t) = Ce^{At}\eta = 9e^t$

$\Rightarrow \eta$ must be the eigenvector for $\lambda_1 = 1$

$0 = [A - I]\eta = \begin{bmatrix} -8 & 8 \\ -4 & 4 \end{bmatrix}\eta$

$\Rightarrow \eta = \gamma \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$y(0) = C\eta = [4 \;\; -1]\begin{bmatrix} \gamma \\ \gamma \end{bmatrix} = 9$

$\Rightarrow 3\gamma = 9 \Rightarrow \gamma = 3$

$\Rightarrow \boxed{\eta = \begin{bmatrix} 3 \\ 3 \end{bmatrix}}$

Q3. 30%

Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for $\mathbb{R}^n$, that is, $v_i^T v_i = 1$ and $v_i^T v_j = 0$ for $i \neq j$. Define $A := v_1 v_2^T + v_2 v_1^T$.

(a) Write range(A).

(b) Write null(A).

(c) Determine the minimal polynomial of A. *Hint:* $A^3 = ?$

(d) Find all the eigenvalues of A.

(e) Find all the eigenvectors of A.

a) $Ax = v_1 \underbrace{v_2^T x}_{\alpha} + v_2 \underbrace{v_1^T x}_{\beta} = \alpha v_1 + \beta v_2$

$\Rightarrow$ range(A) = $\boxed{\text{span}\{v_1, v_2\}}$

b) $Av_i = 0$ for $i = 3, 4, \ldots, n$

$\Rightarrow$ null(A) = $\boxed{\text{span}\{v_3, v_4, \ldots, v_n\}}$

c) $A^2 = (v_1 v_2^T + v_2 v_1^T)(v_1 v_2^T + v_2 v_1^T) = v_1 v_1^T + v_2 v_2^T$

$A^3 = (v_1 v_1^T + v_2 v_2^T)(v_1 v_2^T + v_2 v_1^T) = v_1 v_2^T + v_2 v_1^T = A$

$\Rightarrow A^3 - A = 0 \Rightarrow \boxed{m(s) = s^3 - s = s(s-1)(s+1)}$

d) $\boxed{\lambda(A) = \{0, 1, -1\}}$

e) For $\lambda = 0$

null(A) = span$\{v_3, v_4, \ldots, v_n\}$

$\Rightarrow$ $v_3, v_4, \ldots, v_n$ are the eigenvectors for $\lambda = 0$.

For $\lambda = 1$

$Aw = w$

let $w = \sum_{i=1}^{n} \alpha_i v_i$

$\Rightarrow \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \alpha_i (Av_i) = \alpha_1 Av_1 + \alpha_2 Av_2$

$= \alpha_1 v_2 + \alpha_2 v_1$

That is, $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \alpha_1 v_2 + \alpha_2 v_1$

$\Rightarrow \alpha_1 = \alpha_2$ & $\alpha_i = 0$ for $i \geq 3$

$\Rightarrow w = v_1 + v_2$

For $\lambda = -1$

$Az = -z$ let $z = \sum_{i=1}^{n} \beta_i v_i$

$\Rightarrow -\sum_{i=1}^{n} \beta_i v_i = \sum_{i=1}^{n} \beta_i (Av_i) = \beta_1 v_1 + \beta_2 v_2$

$\Rightarrow \beta_1 = -\beta_2$ & $\beta_i = 0$ for $i \geq 3$

$\Rightarrow z = v_1 - v_2$

Hence all the eigenvectors are:

$\boxed{\{v_1 + v_2, v_1 - v_2, v_3, v_4, \ldots, v_n\}}$

(f) Write the Jordan form of A.

(g) Compute e^{At}.

(h) Compute A^k.

(i) Determine the stability of $\dot{x} = Ax$.

(j) Determine the stability of $x^+ = Ax$.

f) $$\boxed{J = \begin{bmatrix} -1 & & & \\ & 1 & & 0 \\ & & 0 & \\ 0 & & & \ddots \\ & & & & 0 \end{bmatrix}}$$

g) $$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \cdots$$

$$= I + At + \frac{A^2t^2}{2} + \frac{At^3}{3!} + \frac{A^2t^4}{4!} + \cdots$$

$$= I + A\underbrace{\left\{t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right\}}_{\sinh(t)} + A^2\underbrace{\left\{\frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots\right\}}_{\cosh(t)-1}$$

$$\Rightarrow \boxed{e^{At} = I + \sinh(t)\cdot A + (\cosh(t)-1)\cdot A^2}$$

h) $$\boxed{A^k = \begin{cases} I & \text{for } k = 0 \\ A & \text{for } k = 1, 3, 5, \ldots \\ A^2 & \text{for } k = 2, 4, 6, \ldots \end{cases}}$$

i) Due to the eigenvalue at $\lambda = 1$,

$\boxed{\text{unstable}}$

j) All eigenvalues satisfy $|\lambda_i| \leq 1$ &

all the Jordan blocks are 1×1

$\Rightarrow$ $\boxed{\text{stable but not exp. stable}}$

Q4.

Theorem. *Let A, $P \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{m\times n}$. Suppose (C, A) is observable, $P^T = P > 0$, and the Lyapunov equality $A^T PA - P = -C^T C$ holds. Then the system $x^+ = Ax$ is exp. stable.*

Proof. ?

Let $v \in \mathbb{C}^n$ be an eigenvector of A:

$Av = \lambda v$

(C,A) observable $\Rightarrow$ $Cv \neq 0$

$$A^T P A - P = -C^T C$$

$$\Rightarrow v^* (A^T P A - P) v = -v^* C^T C v$$

$$\Rightarrow (Av)^* P Av - v^* P v = -\|Cv\|^2$$

$$\Rightarrow \lambda^* v^* P (\lambda v) - v^* P v = -\|Cv\|^2$$

$$\Rightarrow |\lambda|^2 v^* P v - v^* P v = -\|Cv\|^2$$

$$\Rightarrow (|\lambda|^2 - 1) \underbrace{v^* P v}_{>0} = \underbrace{-\|Cv\|^2}_{<0} \neq 0$$

$$\Rightarrow |\lambda|^2 - 1 < 0$$

$$\Rightarrow |\lambda| < 1 \Rightarrow \text{exp. stable} \quad \blacksquare$$