First name:____________________

Last name: KEY

Student ID:____________________

Signature:____________________

Read before you start:

- There are six questions.
- The examination is open-book.
- No calculator is allowed.
- The duration of the examination is 150 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS.

Q1	Q2	Q3	Q4	Q5	Q6	Total

Q1. 20%

Consider the second order, single-input single-output LTI system below

$$\dot{x} = Ax + \begin{bmatrix} 1 \\ \alpha \end{bmatrix} u$$
$$y = Cx.$$

The following information is known regarding this system.

- $x_a(t) = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$ and $x_b(t) = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$ are two solutions for zero input $u \equiv 0$.
- The system is observable.
- The impulse response is bounded.

(a) Find the eigenvalues and eigenvectors of A. *Hint:* $\dot{x}_a(0) = Ax_a(0)$ *and* $\dot{x}_b(0) = Ax_b(0)$.

(b) Compute A.

(c) Is this system internally stable? Is it BIBO stable?

(d) What is the degree of the transfer function?

(e) Find α.

a) $\dot{x}_a(0) = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = A x_a(0) = A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Likewise $\dot{x}_b(0) = Ax_b(0) \Rightarrow A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Rightarrow \boxed{\lambda_1 = -1, \lambda_2 = 2, v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

b) $A \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$

$= \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}}_{\begin{bmatrix} -2 & 1 \\ -2 & 2 \end{bmatrix}} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}$

$\Rightarrow \boxed{A = \begin{bmatrix} -4 & 3 \\ -6 & 5 \end{bmatrix}}$

c) $\lambda_2 > 0 \Rightarrow$ system is NOT internally stable.

$\left.\begin{array}{l}\text{bounded impulse response} \\ \text{\& no eigenvalues on the imaginary axis}\end{array}\right\} \Rightarrow$ BIBO stable.

d) We must have $\frac{Y(s)}{U(s)} = \frac{\cdots}{(s+1)}$ (bounded impulse response)

$\Rightarrow \boxed{\deg(TF) = 1}$

e) Since $\deg(TF) = 1$ we need pole-zero cancellation. System is observable so it must be uncontrollable. Hence rank$[B \ AB]$ $\Rightarrow$ B is an eigenvector of A. Impulse response is bounded $\Rightarrow$ B is the eigenvector of the stable eigenvalue.

$\Rightarrow B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \boxed{\alpha = 1}$

Q2. 20%

Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = [0 \ \ 1]\, x.$$

(a) Is this system controllable, observable, internally stable?

(b) Can this system be stabilized by static output feedback $u = -ky$?

(c) Find the feedback gain K such that the state feedback $u = -Kx$ places the eigenvalues of the closed-loop system at $\lambda_{1,2} = -1 \pm j$.

(d) Suppose we want to design an observer for this system. Find the observer gain L such that the eigenvalues of the error ($e := \hat{x} - x$ where $\hat{x}$ is the state of our observer) dynamics are at $\lambda_{1,2} = -2, -3$.

a) $\operatorname{rank}[B \ \ AB] = \operatorname{rank}\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = 2 = n$

$\Rightarrow$ $\boxed{\text{controllable}}$

$\operatorname{rank}\begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} = 2 = n$

$\Rightarrow$ $\boxed{\text{observable}}$

$|sI - A| = \begin{vmatrix} s & -1 \\ -3 & s-2 \end{vmatrix} = s(s-2) - 3$
$= s^2 - 2s - 3$

$\Rightarrow \lambda_1 + \lambda_2 = 2 > 0 \Rightarrow \boxed{\text{unstable}}$

b) $\dot{x} = Ax + Bu \big|_{u=-ky} = (A - kBC)x$

$\Rightarrow \underbrace{A - kBC}_{\tilde{A}} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} - k\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 2-k \end{bmatrix}$

$|sI - \tilde{A}| = \begin{vmatrix} s & -1 \\ -3 & s-(2-k) \end{vmatrix} = s^2 - (2-k)s - 3$

$\Rightarrow \forall k \ \ \lambda_1 \cdot \lambda_2 = -3 \Rightarrow$ For all k there is an eigenvalue $\mathrm{Re}\{\lambda_i\} > 0 \Rightarrow \boxed{\text{NO}}$

c) $\underbrace{A - BK}_{\hat{A}} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3-k_1 & 2-k_2 \end{bmatrix}$

$|sI - \hat{A}| = \begin{vmatrix} s & -1 \\ k_1-3 & s+(k_2-2) \end{vmatrix} = s^2 + (k_2-2)s + k_1 - 3$

$\lambda_{1,2} = -1 \mp j \Rightarrow s^2 + 2s + 2 = s^2 + (k_2-2)s + k_1 -$

$\Rightarrow k_2 = 4$ & $k_1 = 5 \Rightarrow \boxed{K = [5 \ \ 4]}$

d) $\underbrace{A - LC}_{\bar{A}} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 0 & l_1 \\ 0 & l_2 \end{bmatrix} = \begin{bmatrix} 0 & 1-l_1 \\ 3 & 2-l_2 \end{bmatrix}$

$\Rightarrow |sI - \bar{A}| = s^2 + (l_2 - 2)s + 3(l_1 - 1)$
$= (s+2)(s+3) = s^2 + 5s + 6$

$\Rightarrow l_2 = 7$ & $l_1 = 3 \Rightarrow \boxed{L = \begin{bmatrix} 3 \\ 7 \end{bmatrix}}$

Q3. 20%

Consider the single-input single-output LTI system below

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned}$$

(5) **(a)** Design the system, i.e., find a suitable (A, B, C) triple, to satisfy the below constraints.

(5) • $C(sI - A)^{-1}B = \dfrac{s+2}{s^2+s+1}$.

(5) • The system is controllable, but *not* detectable.

Hint: You may want to use the observable decomposition here.

(5) **(b)** Determine the unobservable subspace of your design.

a) $\bar{X}_2 := \dfrac{1}{s^2+s+1} U$ & $\bar{X}_1 := s\bar{X}_2$

$\Rightarrow s^2\bar{X}_2 + s\bar{X}_2 + \bar{X}_2 = U \Rightarrow s\bar{X}_1 = -\bar{X}_1 - \bar{X}_2 + U$

$\Rightarrow s\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U$

$Y = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix}$ controllable canonical form

Let $A_o = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$, $B_o = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C_o = \begin{bmatrix} 1 & 2 \end{bmatrix}$

$\Rightarrow$ Then let $A = \left[\begin{array}{c|c} A_o & 0 \\ \hline A_{21} & A_U \end{array}\right]$ $B = \left[\begin{array}{c} B_o \\ \hline 0 \end{array}\right]$

$C = [\, C_o \mid 0 \,]$

Chose $A_U = 0$ & $A_{21} = [0 \;\; 1]$

$$\boxed{\Rightarrow A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}}$$

system NOT detectable because $A_U \geq 0$.
System controllable because it is in cont. canonical form.

b) Since our design is in observable decomposition

$UO = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Q4. 20%

Let v, $w \in \mathbb{R}^n$ be two nonzero vectors that are orthogonal, i.e., $w^T v = 0$. For $A = vw^T$ consider the below system

$$\begin{aligned} \dot{x} &= Ax + vu \\ y &= w^T x. \end{aligned}$$

(a) Write the minimal polynomial of A. *Hint:* $A^2 =$?

(b) Compute e^{At}.

(c) Compute the transfer function.

(d) For the initial condition $x(0) = v$ and constant input $u \equiv 1$ compute the solution $x(t)$.

(e) Compute the controllable subspace.

(f) Can we find an input signal $u(t)$ that transfers $x(t)$ from the origin $x(0) = 0$ to $x(1) = w$?

(g) Is this system detectable?

a) $A^2 = vw^T vw^T = v \cdot 0 \cdot w^T = 0$

$\Rightarrow \boxed{m(s) = s^2}$

b) $e^{At} = I + At + \cancel{\frac{A^2 t^2}{2}} + \cdots$ $\to 0$

$= \boxed{I + vw^T t}$

c) $\dot{y} = w^T \dot{x} = w^T (vw^T x + vu)$

$= \cancel{(w^T v)}^{0} w^T x + \cancel{(w^T v)}^{0} u$

$\Rightarrow \boxed{\frac{Y(s)}{U(s)} = 0}$

d) $x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$

$= (I + vw^T t) v + \int_0^t (I + vw^T (t-\tau)) v \, d\tau$

$= v + vt = \boxed{(t+1)v}$

e) $[B \ AB \ \cdots \ A^{n-1}B] = [v \ 0 \ \cdots \ 0]$

$\Rightarrow \boxed{S_c = \text{span}\{v\}}$

f) No, because $w \notin S_c$.

g) $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} w^T \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\Rightarrow$ system unobservable

since all eigenvalues satisfy $Re\{\lambda_i\} \geq 0$

$\Rightarrow$ system is $\boxed{\text{NOT detectable}}$

Q5. 20%

Consider the single-input single-output LTI system below

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= B^T x. \end{aligned}$$

Suppose that all the eigenvalues of A satisfy $\mathrm{Re}\{\lambda\} \geq 0$ and the below inequality holds

$$A^T + A - BB^T < 0.$$

(a) Show that the system is controllable.

(b) Show that the system is observable.

(c) Show that the system can be stabilized by static output feedback $u = -ky$.

(d) Find the range of k for the closed-loop stability.

a) Choose $P = I$. Then

$PA^T + AP - BB^T < 0 \Rightarrow$ system stabilizable

Since $Re\{\lambda_i\} \geq 0$ for all i, system must be controllable.

b) Choose $P = I$. Then

$A^T P + PA - BB^T < 0 \Rightarrow$ system detectable.

Since $Re\{\lambda_i\} \geq 0 \ \forall i \Rightarrow$ system observable.

c) Closed-loop : $\dot{x} = (A - kBB^T)x$

choose $P = I$.

$$\begin{aligned} 0 &> A^T + A - BB^T \\ &= (A - kBB^T)^T + (A - kBB^T) + (2k-1)BB^T \\ &\geq (A - kBB^T)^T + (A - kBB^T) \quad k \geq \tfrac{1}{2} \\ &= (A - kBB^T)^T P + P(A - kBB^T) \end{aligned}$$

Hence, for $k \geq \frac{1}{2}$ we obtained

$$(A - kBB^T)^T P + P(A - kBB^T) < 0$$

By Lyapunov stability thm we have that $[A - kBB^T]$ is exp. stable.

d) $\boxed{k \geq \frac{1}{2}}$

Q6. 10%

Given a subspace $\mathcal{S} \subset \mathbb{C}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$ let $A\mathcal{S}$ denote the set of all $v \in \mathbb{C}^n$ such that $v = Aw$ for some $w \in \mathcal{S}$. Now consider the LTI system

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx. \end{aligned}$$

Suppose A is nonsingular. Show that if $\mathcal{N}(C) \cap A\mathcal{N}(C) = \{0\}$ then the system is observable.

Suppose not. That is,

$\mathcal{N}(C) \cap A\mathcal{N}(C) = \{0\}$ but system is unobservable. Then

$\exists\ v \neq 0$ such that $Av = \lambda v$
$v \in \mathcal{N}(C)$ (1)

$\Rightarrow A\mathcal{N}(C) \ni Av$
$= \lambda v$

A is nonsingular $\Rightarrow \lambda \neq 0$

$A\mathcal{N}(C)$ subspace, therefore

$\alpha(\lambda v) \in A\mathcal{N}(C)$ for all $\alpha \in \mathbb{C}$

Choose $\alpha = \frac{1}{\lambda} \Rightarrow v \in A\mathcal{N}(C)$ (2)

(1) & (2) $\Rightarrow \mathcal{N}(C) \cap A\mathcal{N}(C) \ni v \neq 0$

Result follows by contradiction. $\blacksquare$