**First name:**____________________________

**Last name:**____________________________

**Student ID:**____________________________

**Signature:**____________________________

**Read before you start:**

- There are six questions.
- The examination is open-book.
- No calculator is allowed.
- The duration of the examination is 150 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS.

| Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Total |
|---|---|---|---|---|---|---|
| | | | | | | |

## Q1. 20%

Consider the second order, single-input single-output LTI system below

$$\begin{aligned} \dot{x} &= Ax + \begin{bmatrix} 1 \\ \alpha \end{bmatrix} u \\ y &= Cx\,. \end{aligned}$$

The following information is known regarding this system.

- $x_{\mathrm{a}}(t) = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$ and $x_{\mathrm{b}}(t) = \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$ are two solutions for zero input $u \equiv 0$.
- The system is observable.
- The impulse response is bounded.

**(a)** Find the eigenvalues and eigenvectors of $A$. *Hint:* $\dot{x}_{\mathrm{a}}(0) = Ax_{\mathrm{a}}(0)$ *and* $\dot{x}_{\mathrm{b}}(0) = Ax_{\mathrm{b}}(0)$.

**(b)** Compute $A$.

**(c)** Is this system internally stable? Is it BIBO stable?

**(d)** What is the degree of the transfer function?

**(e)** Find $\alpha$.

**Q2.** 20%

Consider the system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= [0 \;\; 1]\, x\,.\end{aligned}$$

**(a)** Is this system controllable, observable, internally stable?

**(b)** Can this system be stabilized by static output feedback $u = -ky$?

**(c)** Find the feedback gain $K$ such that the state feedback $u = -Kx$ places the eigenvalues of the closed-loop system at $\lambda_{1,2} = -1 \pm j$.

**(d)** Suppose we want to design an observer for this system. Find the observer gain $L$ such that the eigenvalues of the error ($e := \hat{x} - x$ where $\hat{x}$ is the state of our observer) dynamics are at $\lambda_{1,2} = -2, -3$.

**Q3.** 20%

Consider the single-input single-output LTI system below

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\,.\end{aligned}$$

**(a)** Design the system, i.e., find a suitable $(A,\ B,\ C)$ triple, to satisfy the below constraints.

- $C(sI - A)^{-1}B = \dfrac{s+2}{s^2+s+1}$.
- The system is controllable, but *not* detectable.

*Hint: You may want to use the observable decomposition here.*

**(b)** Determine the unobservable subspace of your design.

**Q4.** 20%

Let $v$, $w \in \mathbb{R}^n$ be two nonzero vectors that are orthogonal, i.e., $w^T v = 0$. For $A = vw^T$ consider the below system

$$\begin{aligned} \dot{x} &= Ax + vu \\ y &= w^T x\,. \end{aligned}$$

**(a)** Write the minimal polynomial of $A$. *Hint:* $A^2 = ?$

**(b)** Compute $e^{At}$.

**(c)** Compute the transfer function.

**(d)** For the initial condition $x(0) = v$ and constant input $u \equiv 1$ compute the solution $x(t)$.

**(e)** Compute the controllable subspace.

**(f)** Can we find an input signal $u(t)$ that transfers $x(t)$ from the origin $x(0) = 0$ to $x(1) = w$?

**(g)** Is this system detectable?

**Q5.** 20%

Consider the single-input single-output LTI system below

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= B^T x \,. \end{aligned}$$

Suppose that all the eigenvalues of $A$ satisfy $\mathrm{Re}\{\lambda\} \geq 0$ and the below inequality holds

$$A^T + A - BB^T < 0 \,.$$

**(a)** Show that the system is controllable.

**(b)** Show that the system is observable.

**(c)** Show that the system can be stabilized by static output feedback $u = -ky$.

**(d)** Find the range of $k$ for the closed-loop stability.

**Q6.** 10%

Given a subspace $\mathcal{S} \subset \mathbb{C}^n$ and a matrix $A \in \mathbb{R}^{n \times n}$ let $A\mathcal{S}$ denote the set of all $v \in \mathbb{C}^n$ such that $v = Aw$ for some $w \in \mathcal{S}$. Now consider the LTI system

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx\,. \end{aligned}$$

Suppose $A$ is nonsingular. Show that if $\mathcal{N}(C) \cap A\mathcal{N}(C) = \{0\}$ then the system is observable.