First name:______________________________

Last name:______________________________

Student ID:______________________________

Section:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 110 minutes.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

For the below 5×5 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

it is known that $\mathbb{C}^5 = \mathcal{N}([A - I]^2) \oplus \mathcal{N}([A - 2I])$ where I is the identity matrix of appropriate size. The aim is to find a transformation matrix P such that $\bar{A} = P^{-1}AP$ is block diagonal of the form

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & \bar{A}_2 \end{bmatrix}$$

where the first block $\bar{A}_1$ is associated with the subspace $\mathcal{N}([A - I]^2)$ and the second block $\bar{A}_2$ with the subspace $\mathcal{N}([A - 2I])$.

(a) Find the sizes of $\bar{A}_1$ and $\bar{A}_2$. (Explain your reasoning.)

(b) Find a transformation matrix P.

(c) Compute $\bar{A}_2$.

Answer:

Q2.

Consider the equation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b\,.$$

Find the solution (or its best approximation) for each of the below cases.

(a) $b = [5 \quad -1 \quad 5]^T$.

(b) $b = [5 \quad 0 \quad -5]^T$.

Answer:

Q3.

A linear transformation $T : \mathbb{C}^3 \to \mathbb{C}^3$ is known to satisfy

$$T\left(\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}\right) = \begin{bmatrix} \alpha + 2\beta \\ \beta + 3\gamma \\ -\gamma \end{bmatrix} \quad \text{for all} \quad \alpha,\, \beta,\, \gamma \in \mathbb{C}$$

(a) Find the matrix representation A of transformation T.

(b) Compute the characteristic polynomial $d(s)$ and the minimal polynomial $m(s)$ of A.

(c) Find the eigenvalues of A.

(d) Using Cayley-Hamilton Theorem compute A^{-1}.

Answer:

Q4.

Let $\mathcal{V}$ be a Hilbert space and $A : \mathcal{V} \to \mathcal{V}$ be a linear transformation with adjoint denoted by A^*.

(a) Show that $(A^*)^* = A$. (Note that A need NOT be a matrix.)

(b) Consider two sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\eta_k\}_{k=0}^{\infty}$ in $\mathcal{V}$. For all $k \geq 0$ suppose that we have $x_{k+1} = A(x_k)$ for the first sequence and $\eta_{k+1} = A^*(\eta_k)$ for the second one. Show that $\langle x_i, \eta_j \rangle = \langle x_j, \eta_i \rangle$ for all $i, j \geq 0$. (For this part *only*, you may assume that $\mathcal{V} = \mathbb{C}^n$ and A is an $n \times n$ matrix.)

Answer: