First name:______________________________

Last name:______________________________

Student ID:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 110 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS!

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

Given a matrix $Q \in \mathbb{R}^{2\times 2}$ we define the function $f_Q : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as $f_Q(x, y) := x^T Q y$. Let

$$R = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) Find a counterexample to show that f_R is NOT an inner product in $\mathbb{R}^2$.

(b) Show that f_P is an inner product in $\mathbb{R}^2$.

(c) Consider the subspace

$$\mathcal{S} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \right\}$$

Let $\mathcal{S}_P^\perp$ and $\mathcal{S}_I^\perp$ denote the orthogonal complements of $\mathcal{S}$ with respect to the inner products f_P and f_I, respectively. Find a suitable value for α so that $\mathcal{S}_P^\perp = \mathcal{S}_I^\perp$.

Q2.

Let $\mathcal{V} = \{f : [0,\, 1] \to \mathbb{R},\ f \text{ square-integrable}\}$ be the Hilbert space of square-integrable functions with the inner product

$$\langle f,\, g\rangle := \int_0^1 f(t)g(t)dt\,.$$

Consider the linear transformation $T : \mathcal{V} \to \mathcal{V}$ defined as

$$T(f) := \int_0^t f(\tau)d\tau\,.$$

Find the adjoint T^* of T.

Q3.

Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Find the vector $\bar{x} \in \mathbb{R}^3$ that simultaneously satisfies:

- $\|A\bar{x} - b\| \leq \|Ax - b\|$ for all $x \in \mathbb{R}^3$.
- $\|A\bar{x} - b\| = \|Ax - b\|$ implies $\|\bar{x}\| \leq \|x\|$.

Q4.

Consider the set of functions $\mathcal{F} = \{f_k\}_{k=0}^{\infty}$ where each $f_k : [0,\, 1] \rightarrow \mathbb{C}$ is defined as

$$\begin{aligned}
f_0(t) &= 1 \\
f_1(t) &= e^{-j2\pi t} \\
f_2(t) &= e^{-j2\pi t} + e^{j2\pi t} \\
f_3(t) &= e^{j2\pi t} + e^{-j4\pi t} \\
f_4(t) &= e^{j4\pi t} + e^{-j6\pi t} \\
&\vdots
\end{aligned}$$

Let the inner product be $\langle f,\, g\rangle := \int_0^1 f(t)g^*(t)dt$ where $g^*(t)$ is the complex conjugate of $g(t)$.

(a) Find the projection of $g(t) = t$ onto the subspace $\mathcal{S}_1 = \text{span}\,\{f_0,\, f_1,\, f_2\}$.

(b) Apply Gram-Schmidt process to $\mathcal{F}$ to obtain an orthogonal set of functions $\mathcal{H} = \{h_k\}_{k=0}^{\infty}$ with span $\mathcal{H} = \text{span}\ \mathcal{F}$.

(c) Find the orthogonal projection of $g(t) = t$ onto the subspace $\mathcal{S}_2 = \text{span}\,\{h_0,\, h_1,\, h_2\}$. Compare the result with the result of part (a).