First name:______________________________

Last name:______________________________

Student ID:______________________________

Section:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 110 minutes.

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

(a) Consider the linear space of 2×2 matrices with complex entries. Show that $\langle A, B\rangle := \mathrm{tr}(B^*A)$ is an inner product, where B^* denotes the conjugate transpose of B and $\mathrm{tr}(B^*A)$ is the trace of B^*A.

(b) Consider the inner product space of 2×2 matrices with the above given inner product. By Gram-Schmidt algorithm compute an orthogonal basis for the subspace below.

$$\mathcal{S} = \mathrm{span}\left\{ \begin{bmatrix} 1 & 2 \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 0 & i \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

Answer:

Q2.

(a) Let function $\nu_{\mathrm{a}} : \mathbb{R}^n \to \mathbb{R}$ satisfy $\nu_{\mathrm{a}}(y) = \max_{\|x\|_1 \le 1} y^T x$. Show that ν_{a} is a norm in $\mathbb{R}^n$.

(b) Let function $\nu_{\mathrm{b}} : \mathbb{R}^n \to \mathbb{R}$ satisfy $\nu_{\mathrm{b}}(y) = \max_{\|x\|_\infty \le 1} y^T x$. Show that ν_{b} is a norm in $\mathbb{R}^n$.

(c) For $y = [2 \quad -3 \quad -6 \quad 8]^T \in \mathbb{R}^4$ compute $\nu_{\mathrm{a}}(y)$ and $\nu_{\mathrm{b}}(y)$.

Answer:

Q3.

Consider the inner product space $\{f \mid f : [0, 1] \to \mathbb{R},\ f \text{ continuous}\}$ with

$$\langle f,\, g\rangle := \int_0^1 f(t)g(t)dt\,.$$

Using Gram-Schmidt algorithm find an orthogonal basis for each of the below subspaces.

(a) $\mathcal{S}_{\mathrm{a}} = \mathrm{span}\{1,\, 1+t,\, 2+t\}$.

(b) $\mathcal{S}_{\mathrm{b}} = \mathrm{span}\{1,\, 1+t,\, 1-t^3\}$.

Answer:

Q4.

Consider the normed space $\mathcal{C}([-1, 1]) = \{f \mid f : [-1, 1] \to \mathbb{R},\ f \text{ continuous}\}$ with

$$\|f\| := \int_{-1}^{1} |f(t)|dt\,.$$

For the sequence of functions $\{f_n\}_{n=1}^{\infty}$ in $\mathcal{C}([-1, 1])$, where $f_n(t) = e^{-n|t|}$, answer the following questions. (Justify your answers.)

(a) Is this a convergent sequence?

(b) Is this a Cauchy sequence?

Answer: