First name:______________________________

Last name:______________________________

Student ID:______________________________

Section:______________________________

Signature:______________________________

Read before you start:

- There are four questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 110 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS!

Q1	**Q2**	**Q3**	**Q4**	**Total**

Q1.

Consider the set of functions $\mathcal{F} = \{f \mid f : \mathbb{R} \to \mathbb{R};\ f(t) = a\cos(t) + b\sin(t);\ a,\ b \in \mathbb{R}\}$ which is a linear space over the field of real numbers. For each of the below cases state whether the candidate is a norm or not. (If your answer is "yes" prove it. Otherwise present a counterexample.)

(a) $\|f\| = a^2 + b^2$

(b) $\|f\| = |a| + |b|$

(c) $\|f\| = |a + b|$

(d) $\|f\| = |a + b| + |b|$

(e) $\|f\| = \max_{t \in \mathbb{R}} |f(t)|$

Q2.

Consider the linear function $\text{tr} : \mathbb{R}^{2\times 2} \to \mathbb{R}$ (which is a mapping from the space of 2×2 real matrices to real numbers) defined as $\text{tr}(A) := a_{11} + a_{22}$ for $A = [a_{ij}]$. Two possible bases $\mathcal{D}_1$, $\mathcal{D}_2$ for the domain ($\mathbb{R}^{2\times 2}$) and two possible bases $\mathcal{R}_1$, $\mathcal{R}_2$ for the range ($\mathbb{R}$) are given below.

$$\begin{aligned} \mathcal{D}_1 &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}, \qquad \mathcal{R}_1 = \{1\}, \\ \mathcal{D}_2 &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}, \qquad \mathcal{R}_2 = \{3\}. \end{aligned}$$

(a) Find all (four) transformation matrices between these bases.

(b) Find the matrix representation of the function $\text{tr}(\cdot)$ with respect to the pair $(\mathcal{D}_1, \mathcal{R}_1)$.

(c) Using change of basis find the matrix representation of $\text{tr}(\cdot)$ with respect to the pair $(\mathcal{D}_2, \mathcal{R}_2)$.

Q3.

Let $\mathcal{P}(N)$ be the linear space of polynomials of degree $\leq N$ over the field of real numbers. It is known that each $p \in \mathcal{P}(2)$ can be written as

$$p(t) = (t+1)q(t) + r(t)$$

where $q \in \mathcal{P}(1)$ and $r \in \mathcal{P}(0)$ are unique. For example, for $p(t) = 3t^2 + 2t + 1$ we have

$$\underbrace{3t^2 + 2t + 1}_{p} = (t+1)(\underbrace{3t - 1}_{q}) + \underbrace{\ 2\ }_{r}.$$

Define transformations $T_1 : \mathcal{P}(2) \to \mathcal{P}(1)$ and $T_2 : \mathcal{P}(2) \to \mathcal{P}(0)$ as $T_1(p) := q$ and $T_2(p) := r$.

(a) Show that T_1 is a linear transformation.

(b) What is $\mathcal{N}(T_1)$? Is T_1 one-to-one?

(c) Choosing suitable bases for the linear spaces, find a matrix representation for T_1.

(d) Show that T_2 is a linear transformation.

(e) What is $\mathcal{N}(T_2)$? Is T_2 one-to-one?

(f) Choosing suitable bases for the linear spaces, find a matrix representation for T_2.

Q4.

(a) Given the sets $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3 \subset \mathbb{R}^3$ below

$$\mathcal{S}_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{S}_2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{S}_3 = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

determine whether the unions $\mathcal{S}_1 \cup \mathcal{S}_2$ and $\mathcal{S}_1 \cup \mathcal{S}_3$ are subspaces or not.

(b) Given an arbitrary linear space $\mathcal{V}$ and two subspaces $\mathcal{Y}$, $\mathcal{Z} \subset \mathcal{V}$ prove that *if* $\mathcal{Y} \cup \mathcal{Z}$ *is a subspace then either* $\mathcal{Y} \subset \mathcal{Z}$ *or* $\mathcal{Z} \subset \mathcal{Y}$.