First name:____________________________

Last name:____________________________

Student ID:____________________________

Signature:____________________________

Read before you start:

- There are five questions.
- The examination is closed-book.
- No calculator is allowed.
- The duration of the examination is 150 minutes.
- PLEASE EXPLAIN ALL YOUR ANSWERS!

Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Total**

Q1.

Let $\mathcal{M}$ denote the linear space of 2×2 real matrices and $\mathcal{P}$ the linear space of polynomials (in t) of degree ≤ 2. Consider the linear mapping $T : \mathcal{M} \to \mathcal{P}$ defined via the following relation

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = bt^2 + (c+d)t + a\,.$$

(a) Choose bases $\mathcal{B}_{\mathrm{m}}$ and $\mathcal{B}_{\mathrm{p}}$ for the spaces $\mathcal{M}$ and $\mathcal{P}$, respectively.

(b) Find the matrix representation of the mapping T with respect to the pair $(\mathcal{B}_{\mathrm{m}},\, \mathcal{B}_{\mathrm{p}})$.

Q2.

Let $\mathcal{V}$ be the real inner product space of polynomials of degree ≤ 2 defined over the interval $[0,\, 1]$ equipped with the inner product

$$\langle p,\, q\rangle := \int_0^1 p(t)q(t)dt\,.$$

Let $\mathcal{S} \subset \mathcal{V}$ be the subspace of constant polynomials (i.e., $p(t) = \text{constant}$). Find a basis for $\mathcal{S}^{\perp}$.

Q3.

Let $\mathcal{V}$ be the normed space of continuous functions defined over the interval $[0,\, 1]$ with the norm

$$\|f\| := \max_{t\in[0,\,1]} |f(t)|\,.$$

Are the following sequences $\{f_n\}_{n=1}^{\infty}$ of functions (in $\mathcal{V}$) convergent when $n \to \infty$? Which function are they converging to if they are convergent?

(a) $f_n(t) = t/n$

(b) $f_n(t) = 1 - t^n$

(c) $f_n(t) = \sin(nt)$

(d) $f_n(t) = \sin(t/n)$

Q4.

Consider the following matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

It is known that all the eigenvalues of this matrix are at $\lambda = 1$.

(a) Find the minimal polynomial of A.

(b) Find A^{-1}.

(c) Find the Jordan form J of the matrix A. (Assume that the blocks are ordered from largest to smallest.)

(d) Find the matrix P that transforms A into J, i.e., $P^{-1}AP = J$.

Q5.

Let $v \in \mathbb{R}^n$ be a unit vector (i.e., $\langle v, v \rangle = v^T v = 1$). Define the $n \times n$ matrix $P := vv^T$.

(a) Show that P is a projection matrix. In other words, prove that $P^2 = P$ and $P^T = P$.

(b) Show that the minimal polynomial of P is $m(s) = s(s-1)$. *Hint:* $P^2 = P$.

(c) What are the eigenvalues of P?

(d) Find $\mathcal{R}(P)$, rank of P, and $\dim \mathcal{N}(P)$.

(e) Write the characteristic polynomial $d(s)$ of P.

(f) Write the subspace $\mathcal{S} \subset \mathbb{R}^n$ that P projects onto.

(g) How many linearly independent eigenvectors does P have?

(h) Write the Jordan form of P.