Q1. In each part below some information is given regarding a matrix with characteristic polynomial $d(s) = (s-2)^{10}$. Find possible Jordan forms of that matrix.

(a) $m(s) = (s-2)^3$, $\text{rank}\,(A-2I) = 6$, $\text{rank}\,(A-2I)^2 = 3$.

(b) $m(s) = (s-2)^4$, $\dim \mathcal{N}(A-2I) = 3$, $\dim \mathcal{N}(A-2I)^2 = 6$, $\dim \mathcal{N}(A-2I)^3 = 8$.

Q2. Obtain the Jordan form and the transformation matrix for the below matrix.

$$\begin{bmatrix} 2 & 2 & 2 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

Q3. Obtain the Jordan form for the below matrices.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Q4. Let $A \in \mathbb{C}^{n\times n}$ have $d(s) = s^k(s-\alpha)^{n-k}$ and $m(s) = s^\ell(s-\alpha)$. Determine $\dim \mathcal{N}(A^\ell)$ and $\dim \mathcal{N}(A-\alpha I)$.

Q5. Given $A \in \mathbb{C}^{n\times n}$, let λ_i be an eigenvalue of A. It is known that $\dim \mathcal{N}(A-\lambda_i I) = 2$, $\dim \mathcal{N}(A-\lambda_i I)^2 = 4$, $\dim \mathcal{N}(A-\lambda_i I)^3 = 5$, and $\dim \mathcal{N}(A-\lambda_i I)^4 = \dim \mathcal{N}(A-\lambda_i I)^5 = 6$. Find the Jordan block corresponding to λ_i.

Q6. Consider the polynomials $d_1(s) = (s-1)^2(s-2)^2$ and $d_2(s) = (s^2+1)^2$. One of these polynomials is known to be the characteristic polynomial of a Hermitian matrix A (Can you tell which one?) Determine *the* Jordan form of A.

Q7. Let $\omega > 0$. For

$$A = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

find e^{At} and $\cos(At)$.

Q8. For each of the below A matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

find a second order polynomial $r(s)$ such that $r(A) = e^A$.

Q9. Let A be a nonsingular Hermitian matrix. Prove the following.

(a) A^{-1} is Hermitian.

(b) If A is positive definite then A^{-1} is positive definite, too.

Q10. Given two vectors u, $v \in \mathbb{C}^n$, find the Jordan form of the matrix $A = uv^*$.

Q11.[1] Let

$$u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad r = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.4 \end{bmatrix}$$

Design a 4-by-4 matrix G simultaneously satisfying the below conditions.

- $G^k = ur^T$ for all $k \geq 3$.
- $G^2 \neq ur^T$.

Show that $[\alpha G + (1-\alpha)ur^T]^k = ur^T$ for all $\alpha \in \mathbb{R}$ and $k \geq 3$.

Q12. Show that there cannot exist a 3-by-3 matrix A satisfying the below equality.

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Q13.[2] Let Q and R be two Hermitian positive definite matrices with $Q^2 = R^2$. Prove $Q = R$.

Q14. Let $A \in \mathbb{C}^{n \times n}$. Prove the following.

(a) If each $v \in \mathbb{C}^n$ is an eigenvector of $A \in \mathbb{C}^{n \times n}$, then $A = \lambda I$ for some $\lambda \in \mathbb{C}$.

(b) Suppose A^{-1} exists. Then $\lambda \in \mathbb{C}$ is an eigenvalue of A if λ^{-1} is an eigenvalue of A^{-1}.

(c) For each $k = 1, 2, \ldots, n$ we can find a subspace $\mathcal{U}_k$ invariant under A with $\dim \mathcal{U}_k = k$.

Q15. Let A and B be arbitrary matrices in $\mathbb{C}^{n \times n}$. Prove or disprove the below claims.

(a) If $e^{At} = e^{Bt}$ for all $t \in \mathbb{R}$ then $A = B$.

(b) $e^A = e^B$ implies $A = B$.

(c) $(e^A)^{-1}$ exists.

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[1] This problem may be difficult. Feel free to use MATLAB.

[2] This problem may be difficult.