EE501 HW3

Q1. Let $n \times n$ matrix A have n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Given some polynomial $p(s)$, show that the eigenvalues of matrix $p(A)$ are $p(\lambda_1), p(\lambda_2), \ldots, p(\lambda_n)$.

Q2. Consider the standard inner product $\langle x, y\rangle := y^*x$ in $\mathbb{C}^n$. Let $\|x\| = \langle x, x\rangle^{1/2}$ for $x \in \mathbb{C}^n$ and $\|A\| = \max_{\|x\|=1} \|Ax\|$ for $A \in \mathbb{C}^{n\times n}$. Show that $\|A\|^2$ equals the largest eigenvalue of A^*A. *Hint: The eigenvalues of A^*A are real and nonnegative. Moreover, A^*A has n eigenvectors that are pairwise orthogonal.*

Q3. Let $\mathbb{C}^n = \mathcal{U} \oplus \mathcal{V}$ where subspace $\mathcal{U}$ is known to be invariant under some linear transformation $A : \mathbb{C}^n \to \mathbb{C}^n$. Show that with respect to a basis formed by concatenating bases for $\mathcal{U}$ and $\mathcal{V}$, mapping A has a matrix representation

$$\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$$

Determine also the size of each block in this representation.

Q4. Let A be a 2×2 matrix with $A^2 = 0$. Prove that either one of the following must be true.

- $A = 0$.
- There exists P such that

$$P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Q5. Let A be a 2×2 matrix. Prove that there exists P such that either

$$P^{-1}AP = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \quad \text{or} \quad P^{-1}AP = \begin{bmatrix} \alpha & 0 \\ 1 & \alpha \end{bmatrix}$$

Hint: See Q4.

Q6. Let A be an $n \times n$ matrix with $A^k = 0$ for some integer k. Prove that $A^n = 0$.

Q7. Find the characteristic polynomial $d(s)$ and minimal polynomial $m(s)$ for the below matrices.

$$\begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 1 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 1 & \lambda_1 & 0 & 0 & 0 \\ 0 & 1 & \lambda_1 & 0 & 0 \\ 0 & 0 & 1 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 \end{bmatrix}, \quad \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 1 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 1 & \lambda_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 \end{bmatrix}$$

Q8. Let $v \in \mathbb{C}^n$ be a nonzero vector.

(a) Show that $P = \dfrac{vv^*}{v^*v}$ is an orthogonal projection matrix. What is the subspace that P projects onto?

(b) Find $d(s)$ and $m(s)$ for P.

Q9. Consider Hilbert space $\mathcal{V} = \{f | f : [0, 2\pi] \to \mathbb{C}, f \text{ continuous}\}$ with inner product $\langle f, g \rangle := \int_0^{2\pi} f(t)\overline{g(t)}dt$. Let $f_1, f_2, f_3 \in \mathcal{V}$ satisfy $f_1(t) = 1$, $f_2(t) = t$, and $f_3(t) = e^{jt}$ for all t. Define $\mathcal{M} := \text{span}\{f_2, f_3\}$. For which $g \in \mathcal{M}$ is $\|f_1 - g\|$ minimum?

Q10. Consider Hilbert space $\mathcal{V} = \{f | f : [-\pi, \pi] \to \mathbb{C}, f \text{ continuous}\}$ with inner product $\langle f, g \rangle := \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$. Define $\mathcal{M} := \text{span}\{1, \sin t, \cos t, e^{jt}, e^{-jt}\}$.

(a) Determine $\dim M$.

(b) Find the orthogonal projections of the following functions onto $\mathcal{M}$: $f_1(t) = e^t$, $f_2(t) = 5$, $f_3(t) = e^{-j2t}$.

Q11. Consider the space of 2×2 matrices with $\langle A, B \rangle := \text{tr}(B^*A)$.

(a) Find a basis for $\mathcal{M}^\perp$ where

$$\mathcal{M} = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & j \end{bmatrix}\right\}$$

(b) Find the orthogonal projections of the below matrices onto $\mathcal{M}$.

$$\begin{bmatrix} 5j & 0 \\ 0 & 5 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Q12. Consider the linear systems of equations

$$\begin{bmatrix} 3 & 4 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 14 \\ 10 \\ 21 \\ 6 \\ 2 \end{bmatrix}$$

(a) Comment on the existence and uniqueness of the solution.

(b) Find the exact (or approximate) solution.

Q13. Consider

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b\,.$$

Find the solution (or approximate solution) for $b = [5 \ \ -1 \ \ 5]^T$ and $b = [5 \ \ 0 \ \ -5]^T$.

Q14. Let

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [3 \ \ 2 \ \ 1]$$

(a) Find bases for $\mathcal{R}(A)$ and $\mathcal{R}(A^T)$.

(b) Find bases for $\mathcal{N}(A)$ and $\mathcal{N}(A^T)$.

(c) Show that $\mathcal{R}(A) \perp \mathcal{N}(A^T)$.

Q15. Consider equation $Ax = b$, where $A = vv^*$ for some $v \in \mathbb{C}^n$.

(a) Find $\mathcal{R}(A)$ and $\mathcal{N}(A)$.

(b) Discuss the existence and uniqueness of the solution.

Q16. Note that matrix $Q \in \mathbb{R}^{n\times n}$ is said to be *orthogonal* if $Q^TQ = I$ and that matrix $S \in \mathbb{R}^{n\times n}$ is said to be *skew-symmetric* if $S + S^T = 0$. Prove the following.

(a) If $\lambda \in \mathbb{C}$ is an eigenvalue of an orthogonal matrix then $|\lambda| = 1$.

(b) If $\lambda \in \mathbb{C}$ is an eigenvalue of a skew-symmetric matrix then $\mathrm{Re}(\lambda) = 0$.

(c) If S is skew-symmetric then e^S is orthogonal, where $e^S = I + S + \dfrac{S^2}{2} + \dfrac{S^3}{3!} + \dots$

Q17. Find $d(s)$ and $m(s)$ for the matrix below. *Hint: The matrix has a single eigenvalue.*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$