Lecture Notes for Linear System Theory

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JORDAN CANNONICAL FORM

We have discussed the problem of representing a transformation (defined by the matrix A) with a matrix in as simple as possible form. In other words the problem is to find a basis P to C^n such that $P^{-1}AP$ is of simple form.

We have shown that, if A has all eigenvalues distinct then we can find a matrix $P = [e_1 e_2 \cdots e_n]$ where e_i is an eigenvector associated with λ_i such that $P^{-1}AP$ is in diagonal form. But in general this can not be achieved for all A. But we can find a basis with respect to which the representation $P^{-1}AP$ is of relatively simple form called Jordan Cannonical Form.

In the following we shall discuss what this relatively simple form is and how to find the basis to achieve this Jordan Form.

Theorem (Jordan Cannonical Form)
Suppose A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_\sigma$ each of multiplicity m_i in the minimal polynomial and d_i in the characteristic polynomial. Then there exists an $n \times n$ matrix P (columns of P are basis for C^n) called ~~the~~ modal matrix such that

$$P^{-1}AP = J = \begin{pmatrix} J_1 & & & O \\ & J_2 & & \\ & & \ddots & \\ O & & & J_\sigma \end{pmatrix}$$

Here J_i is a $d_i \times d_i$ matrix such that

$$J_i = \begin{pmatrix} J_{i,1} & & \bigcirc \\ & \ddots & \\ \bigcirc & & J_{i,k_i} \end{pmatrix}$$

$$J_{i,1} = \begin{pmatrix} \lambda_i & 1 & & \bigcirc \\ 0 & \lambda_i & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ & \bigcirc & & \lambda_i \end{pmatrix}$$

$J_{i,1}$ is the largest block $m_i \times m_i$ matrix. $J_{i,2}$ and $J_{i,3}$ and others are of the same type as $J_{i,1}$ with dimensions <u>non-increasing</u> such that the over all dimension of J_i is d_i.

The sizes of these matrices can precisely be given as follows. Define

$$\gamma_0 \triangleq \dim N(A - \lambda_i I)$$

$$\gamma_j \triangleq \dim\left[N(A - \lambda_i I)^{j+1}\right] - \dim\left[N(A - \lambda_i I)\right)$$

$$j = 1, 2, \dots, m_i - 1$$

It can be shown that

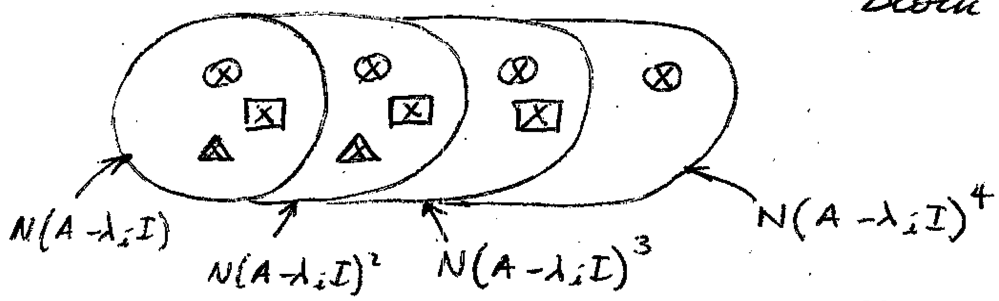
$$\gamma_0 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{m_i - 1}.$$

Size of $J_{i,2}$ is the largest integer k for which $\gamma_{k-1} - 1$ is positive.

$\vdots$

Size of $J_{i,\ell}$ is the largest integer k for which $(\gamma_{k-1} - \ell + 1)$ is positive, $\ell = 2, \dots, k_i$

<u>Example</u> : An alternate approach to determine the block sizes.



$m_i = 4$

$d_i = 9$

Determine,

$$\dim N(A-\lambda_i I) = 3$$
$$\dim N(A-\lambda_i I)^2 = 6$$
$$\dim N(A-\lambda_i I)^3 = 8$$
$$\dim N(A-\lambda_i I)^4 = 9$$

$J_{i,1}$ is 4×4 since $m_i = 4$ $\circ$

$J_{i,2}$ is 3×3, shown by $\square$

$J_{i,3}$ is 2×2, shown by $\triangle$

<u>Example</u> : For

$$A = \begin{pmatrix} 3 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

a modal matrix P exists

$$P = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

for which

$$P^{-1}AP = J = \left(\begin{array}{ccc|cc|c} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

<u>Example</u>

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\det(sI - A) = s^5$$

$\lambda = 0$ is the only eigenvalue, has multiplicity 5 in the characteristic polynomial. i.e.

$$d_1 = 5$$

<u>How to Find m_i?</u> There are two ways.

a) Starting from smallest powers of each factor $(\lambda - \lambda_i)$ substitute A instead of λ and determine the smallest powers for which we get 0 matrix.

e.g. in the above case try $(A - \overset{0}{\lambda} I) = A \overset{?}{=} 0$

$$A^2 \overset{?}{=} 0$$

$$A^3 = 0.$$

So $m_1 = 3$

When there are more distinct eigenvalues m_i should be found simultaneously!

b) Determine $\dim N(A - \lambda_i I)$, then

$\dim N(A - \lambda_i I)^2$, then

$\dim N(A - \lambda_i I)^3$ and so on

until $\dim N(A-\lambda_i I)^{k_i} = N(A-\lambda_i I)^{k_i+1}$
then $k_i = m_i$. //.

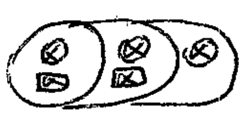
Check for smallest k for which
$$(A-\lambda I)^k = 0 \qquad \lambda = 0$$

$$A \neq 0$$
$$A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \neq 0$$

$$A^3 = 0$$
Therefore $m_1 = 3$.
Then, determine

$\dim N(A - 0I) = \gamma_0 = 2$
$\dim N(A - 0I)^2 = \gamma_0 + \gamma_1 = 4$
$\dim N(A - 0I)^3 = \gamma_0 + \gamma_1 + \gamma_2 = 5$
$\dim N(A - 0I)^4 = 5$

hence, again we see that $m_1 = 3$
Now to determine the size of the smaller blocks



$J_{i,1} = 3$
$J_{i,2} = 2$

Size of $J_{1,1}$ is $m_1 = 3$.
To find size of $J_{1,2}$ we apply theorem.

$k=3 \qquad \gamma_2 - 1 = 1-1 = 0$ not positive
$k=2 \qquad \gamma_1 - 1 = 2-1 = 1 > 0$
$k=1 \qquad \gamma_0 - 1 = 2-1 = 1$
largest k for which $\gamma_{k-1} - 1$ is positive
$k=2$. So size of $J_{1,2} = 2$

$\gamma_{k-1} - \underset{\|}{\ell} + 1$ will not give any positive number

So J_1 block is completed. Also see this from $J_{1,1} + J_{1,2} = 3 + 2 = 5 = d_1$

So we do not look for another block $J_{1,3}$. Also there is no J_2 or J_3 since there is just one distinct eigenvalue.

So

$$P^{-1}AP = J = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & 0 & & \\ \hline & 0 & & 0 & 1 \\ & & & 0 & 0 \end{array}\right) \quad .//.$$

As you see J jordan form representation matrix can be found without actually finding P the modal matrix. But we may need to find P anyway so in the following we shall give a procedure to find the modal matrix P.

MODAL MATRIX P

We shall give two algorithms to find the modal matrix P. In the first we assume that we have already obtained a block diagonal representation, we shall give an algorithm that will start from this point. The second method is more direct in the sense that we obtain it directly from A. The second method is more economical if we have not already obtained a block diagonal representation.

1) Modal Matrix P from a Block Diagonal representation.

Suppose the $n \times n$ matrix A has $\lambda_1, \lambda_2, \ldots, \lambda_\sigma$ as its distinct eigenvalues with multiplicities d_i in the characteristic polynomial and m_i in the minimal polynomial.

Recall that

$$C^n = N(A-\lambda_1 I)^{m_1} \oplus \cdots \oplus N(A-\lambda_\sigma I)^{m_\sigma}$$

Let $\{ b_i^j \}_{j=1}^{d_i}$ be any basis for $N(A-\lambda_i I$ given or determined. Define

$$B_i \triangleq (b_i^1 \mid b_i^2 \mid \cdots \mid b_i^{d_i})$$

$$B \triangleq (B_1 \mid B_2 \mid \cdots \mid B_\sigma)$$

Then we have proved that

$$B^{-1} A B = \bar{A} = \begin{pmatrix} \bar{A}_1 & & 0 \\ & \ddots & \\ 0 & & \bar{A}_\sigma \end{pmatrix}$$

$\bar{A}$ is in block diagonal form, but not necessarily in J (Jordan) form. To transform into J we need to find another basis for C^n.

For each i we do the following:

- Determine m_i
- Determine the size of each Jordan sub-blocks $J_{i,l}$.

- Define $M_i \triangleq (\bar{A}_i - \lambda_i I)$
a $d_i \times d_i$ matrix.

- Find $b_{i,1}$ such that $M_i^{m_i-1} b_{i,1} \neq \theta$
($M_i^{m_i} b_{i,1} = \theta$ since $M_i^{m_i} = 0$ as proved)

- Then form the <u>chain</u> of size m_i
$(M_i^{m_i-1} b_{i,1}, M_i^{m_i-2} b_{i,1}, \dots, M_i b_{i,1}, b_{i,1}$

- Note the size of the next biggest Jordan sub-block, say it is $\underline{q}$.

- Find a vector $b_{i,2}$ such that

$$M_i^{q} b_{i,2} = 0 \quad \text{and}$$

$$\{M_i^{m_i-1} b_{i,1}, M_i^{q_i-1} b_{i,2}\}$$

are linearly independent.

- Then form the chain

$$(M_i^{q-1} b_{i,2}, \dots, M_i b_{i,2}, b_{i,2})$$

- Look for the size of the next biggest jordan subblock, say it is r.

- Find a vector $b_{i,3}$ such that

$$M_i^{r} b_{i,3} = \theta \quad \text{and}$$

$$\{M_i^{m_i-1} b_{i,1}, M_i^{q-1} b_{i,2}, M_i^{r-1} b_{i,3}\}$$

are linearly independent

- Form the chain

$$(M_i^{r-1} b_{i,3}, \dots, M_i b_{i,3}, b_{i,3})$$

- And obtain all such chains, then form

$$\bar{B}_i = (M_i^{m_i-1} b_{i,1}, \dots, b_{i,1}, M_i^{q-1} b_{i,2}, \dots, b_{i,2}, \dots)$$

Then

$$\bar{B}_i^{-1} \bar{A}_i \bar{B}_i = J_i$$

Then

$$\bar{B} \triangleq \begin{pmatrix} \boxed{\bar{B}_1} & & O \\ & \boxed{\bar{B}_2} & \\ O & & \ddots & \\ & & & \boxed{\bar{B}_\sigma} \end{pmatrix}$$

$$\bar{B}^{-1} \underbrace{B^{-1} A B}_{\bar{A}} \bar{B} = J$$

So the modal matrix $P = B\bar{B}$. //.

We need to prove two things.

1. ~~The columns of $\bar{B}_i$ is a basis to $N(A-\lambda_i I)^{m_i}$~~ ~~i.e.~~ they are linearly independent
2. $\bar{B}_i^{-1} \bar{A}_i \bar{B}_i = J_i$.

<u>Proof of 1</u> Consider

$$\alpha_1 M_i^{m_i-1} b_{i,1} + \alpha_2 M_i^{m_i-2} b_{i,1} + \dots + \alpha_{d_i} b_{i,e} = \theta$$

Multiplying from the left by various powers of M_i and using the facts in constructing $b_{i,j}$'s we can successively show that $\alpha_i = 0$ for all i. Hence the d_i columns are linearly independent, hence a basis.

[illegible]

Proof of 2: $\bar{A}_i \bar{B}_i \stackrel{?}{=} \bar{B}_i J_i$

$\bar{A}_i = M_i + \lambda_i I$ So equivalently

$$\bar{B}_i^{-1} M_i \bar{B}_i \stackrel{?}{=} J_i - \lambda_i I = K_i = \begin{pmatrix} \begin{matrix} 0 & 1 & \\ & \ddots & \ddots \\ 0 & & 0 \end{matrix} & 0 \\ 0 & \ddots \end{pmatrix}$$

$$M_i \bar{B}_i \stackrel{?}{=} \bar{B}_i K_i$$

$$M_i \bar{B}_i = (0 \; M_i^{m-1} b_{i,1}, \ldots, M_i b_{i,1}, 0, M_i^{q-1} b_{i,2}, \ldots, M_i b_{i,2}, \ldots)$$

$$\bar{B}_i K_i = (0 \; M_i^{m_i-1} b_{i,1}, \ldots, M_i b_{i,1}, 0, M_i^{q-2} b_{i,2}, \ldots)$$

QED.

Special case: $m_i = d_i$

Then J_i consists of only one subblock, and consequently there is one chain.

$$\bar{B}_i = (M_i^{m_i-1} b_{i,1}, M_i^{m_i-2} b_{i,1}, \ldots, M_i b_{i,1}, b_{i,1})$$

$$\bar{B}_i^{-1} \bar{A}_i \bar{B}_i = J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$$

2) Modal Matrix P directly from A. We prefer this method because it is done in one step.

Suppose the $n \times n$ matrix A has $\lambda_1, \lambda_2, \ldots, \lambda_\sigma$ as its distinct eigenvalues with multiplicities d_i in the characteristic polynomial and m_i in the minimal polynomial.

We would like to find bases for $N(A - \lambda_i I)^{m_i}$ and a basis for C^n (i.e. matrix P) such that

$$P^{-1} A P = J.$$

- For each i do the following:
- Determine m_i and the sizes of $J_{i,\ell}$
- Find $b_{i,1}$ such that

$$(A - \lambda_i I)^{m_i} b_{i,1} = \theta$$

$$(A - \lambda_i I)^{m_i - 1} b_{i,1} \neq \theta$$

Then form the <u>chain</u>

$$\left((A-\lambda_i I)^{m_i-1} b_{i,1}, \ldots, b_{i,1} \right)$$

- Let q be the size of $J_{i,2}$, then find $b_{i,2}$ such that $(A - \lambda_i I)^q b_{i,2} = \theta$

and $\left\{ (A-\lambda_i I)^{m_i-1} b_{i,1}, (A-\lambda_i I)^{q-1} b_{i,2} \right\}$ are linearly independent

then form the chain

$$\left((A-\lambda_i I)^{q-1} b_{i,2}, \ldots, (A-\lambda_i I) b_{i,2}, b_{i,2} \right)$$

- Let r be the size of $J_{i,3}$ then find $b_{i,3}$ such that $(A-\lambda_i I)^r b_{i,3} = \theta$ and $\left\{ (A-\lambda_i I)^{m_i-1} b_{i,1}, (A-\lambda_i I)^{q-1} b_{i,2}, (A-\lambda_i I)^{r-1} b_{i,3} \right\}$ are linearly independent.

Form the chain

$$\left((A-\lambda_i I)^{r-1} b_{i,3}, \ldots, b_{i,3} \right)$$

- And obtain all such chains. Then attach all of them in decreasing size to form

$$P_i = \left((A-\lambda_i I)^{m_i-1} b_{i,1}, \ldots, b_{i,1}, (A-\lambda_i I)^{q-1} b_{i,2}, \ldots, b_{i,2}, \ldots \right)$$

- Then form the modal matrix

$$P = \left(P_1 \mid P_2 \mid \cdots \mid P_\sigma \right)$$

– Then $P^{-1}AP = J$.

We need to prove that:

(i) columns of P_i is a basis for $N(A-\lambda_i I)^{m_i}$

(ii) $P^{-1}AP = J$ (existance of $b_{i,j}$ vectors need a proof)

proof:

(i) To show that columns of P_i is a basis for $N(A-\lambda_i I)$ is equivalent to showing the columns of P_i are in $N(A-\lambda_i I)^{m_i}$ and that they are linearly independent since P_i has d_i columns and $\dim N(A-\lambda_i I)^{m_i} = d_i$

Consider an arbitrary column of P_i

$$(A-\lambda_i I)^{\ell}\, b_{i,\cancel{j}} \qquad \ell = 0,1,\dots, \underset{\text{chain}}{\overset{k_j}{\text{size of } j^{th}}}-1$$

a vector in the $\cancel{j}^{th}$ chain. ~~Suppose~~ Consider

$$(A-\lambda_i I)^{m_i}(A-\lambda_i I)^{\ell}\, b_{i,\cancel{j}} = (A-\lambda_i I)^{m_i+\ell}\, b_{i,\cancel{j}} = 0$$

Since $m_i + \ell \geq$ size of the j^{th} chain $(\cancel{k_j})$ and since $b_{i,\cancel{j}}$ satisfies $(A-\lambda_i I)^{\cancel{k_j}} b_{i,\cancel{j}} = 0$

Hence each column of P_i is in $N(A-\lambda_i I)^{m_i}$.

To show linear independence

$$\alpha_1 (A-\lambda_i I)^{m_i-1} b_{i,1} + \dots + \alpha_{m_i} b_{i,1} + \dots + \alpha_{d_i} b_{i,\cancel{k}_i} = \theta$$

Multiplying from the left by $(A-\lambda_i I)^{m_i-1}$, $(A-\lambda_i I)^{m_i-2}$, ··· successively, each time implying an $\alpha_j = 0$ because of the conditions satisfied by $b_{i,\cancel{j}}$. As a result we obtain all $\alpha_j = 0$. Hence the columns of P_i are linearly independent

(ii) $P^{-1}AP \stackrel{?}{=} J$

i.e $AP \stackrel{?}{=} PJ$

i.e $AP_i \stackrel{?}{=} P_i J_i$

or $(A-\lambda_i I)P_i \stackrel{?}{=} P_i(J_i-\lambda_i I)$

$$(A-\lambda_i I)P_i = (0, (A-\lambda_i I)^{m_i-1} b_{i,1}, \ldots, (A-\lambda_i I) b_{i,1}, 0, \ldots)$$

$$P_i(J_i-\lambda_i I) = (0, (A-\lambda_i I)^{m_i-1} b_{i,1}, \ldots, (A-\lambda_i I) b_{i,1}, 0, \ldots)$$

QED.

Example: For

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\lambda = 0$ is the only eigenvalues because

$\det(sI-A) = s^5$

We have obtained that

$$\left.\begin{array}{l} d_1 = 5 \\ m_1 = 3 \\ \text{size of } J_{1,2} = 2 \end{array}\right\} \qquad J = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & & \\ 0 & 0 & 1 & \multicolumn{2}{c}{0} \\ 0 & 0 & 0 & & \\ \hline \multicolumn{3}{c|}{0} & 0 & 1 \\ & & & 0 & 0 \end{array}\right)$$

Now to find P such that $P^{-1}AP = J$

Find $b_{1,1}$ such that $(A-\lambda_1 I)^{m_1-1} b_{1,1} \neq \theta$

and $(A-\lambda_1 I)^{m_1} b_{1,1} = \theta$

That is to find $b_{1,1}$ such that

$$A^3 b_{1,1} = \theta \quad \text{and} \quad A^2 b_{1,1} \neq \theta$$

$A^3 = O$ so pick any vector as $b_{1,1}$ for which $A^2 b_{1,1} \neq \theta$

$$b_{1,1} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

chain $\quad (A^2 b_{1,1}, A b_{1,1}, b_{1,1})$

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

$\underline{q} = 2$

Find $b_{1,2}$ such that $\quad A^2 b_{1,2} = \theta$

and $\{A^2 b_{1,1}, A b_{1,2}\}$ are l.i.

Look for the basis vectors of $N(A^2)$. Pick one so if the l.i. condition is satisfied. If not pick another basis vector which satisfies it.

$$b_{1,2} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \qquad A^2 b_{1,2} = 0$$

$$A b_{1,2} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } A^2 b_{1,1} \text{ are l.i.}$$

So

Chain. $\quad (A b_{1,2}, b_{1,2})$

$$\left(\begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ \vdots \end{pmatrix} \right)$$

So

$$P = \begin{pmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and $AP = PJ$, Checks.

Example: Try the example on page 36.

Theorem:

a) If matrix A is symmetric then $m_i = 1$ for all i.

b) If matrix A is in companion form, i.e.

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ & & & & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{pmatrix}.$$

then

$$m_i = d_i \qquad i = 1, 2, \ldots, \sigma$$

And $\begin{pmatrix} 1 \\ \lambda_i \\ \lambda_i^2 \\ \vdots \\ \lambda_i^{n-1} \end{pmatrix}$ is an eigenvector associated with λ_i

And

$$\left(\begin{pmatrix} 1 \\ \lambda_i \\ \lambda_i^2 \\ \vdots \\ \lambda_i^{n-1} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2\lambda_i \\ \\ (n-1)\lambda_i^{n-2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \\ \binom{n-1}{2}\lambda_i^{n-3} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \binom{n-1}{3}\lambda_i^{n-4} \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \binom{n-1}{m_i}\lambda_i^{n-m_i} \end{pmatrix} \right),$$

is a chain of generalized eigenvectors, i.e basis for $N(A - \lambda_i I)^{m_i}$, associated with λ_i.

e) If you interchange places of chains in P51 then the corresponding blocks will change in J. If you reverse the order of the columns in P then the order of blocks will change in J and furthermore the 1's appear in the below diagonal.

MODAL INTERPRETATION

Consider the homogenous system

$$\dot{x}(t) = A\,x(t)$$

and let M be an invariant (under A) subspace of C^n. Then if the initial state x_0 is in M then

$$x(t) = s(t, 0, x_0, \theta_u) = e^{At} x_0 \in M \qquad \forall t \geq 0$$

Proof:

$$e^{At} \triangleq \sum_{i=0}^{\infty} \frac{A^i t^i}{i!}$$

$$e^{At} x_0 = \sum_{i=0}^{\infty} \frac{t^i}{i!} (A^i x_0)$$

For $x_0 \in M$, $A x_0 \in M$ since M is invariant
$A^2 x_0 \in M$ " " " "
$\vdots$

Since $e^{At} x_0$ is a linear combination of the vectors $x_0, A x_0, A^2 x_0, \ldots$ all in M and M is a subspace (we need closed subspace)
Hence,

$$e^{At} x_0 \in M \qquad .//.$$

Also using Cayley-Hamilton can be shown without needing closedness.