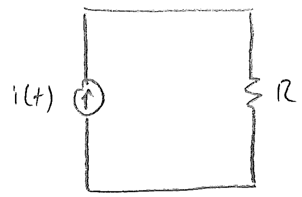
 26 Aralık 2023

Effective (RMS) value

Consider two circuits

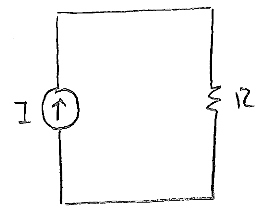


N_{AC}

$i(t+T) = i(t)$, periodic signal

instantaneous power dissipated on the resistor : $p(t) = R\, i(t)^2$

average power : $P_{av} = \frac{1}{T}\int_0^T p(t)dt$



N_{DC}

$I > 0$, constant

$p(t) = P_{av} = R I^2$

Suppose : The average powers are equal for N_{AC} and N_{DC}.

Question : What does the above condition imply?

$$\Rightarrow \frac{1}{T}\int_0^T R\, i(t)^2 dt = R I^2 \quad \Rightarrow \quad \frac{1}{T}\int_0^T i(t)^2 dt = I^2$$

$$\Rightarrow \sqrt{\frac{1}{T}\int_0^T i(t)^2 dt} = I$$

This motivates the following definition.

Definition Given a periodic signal $i(t) = i(t+T)$ the effective or RMS (root mean square) value $I_{eff} > 0$ is defined via

$$I_{eff} = \left[\frac{1}{T}\int_0^T i(t)^2 dt\right]^{1/2}$$

Remark For $i(t) = A\cos(\omega t + \phi)$ $(A>0)$ we have $I_{eff} = \frac{A}{\sqrt{2}}$.

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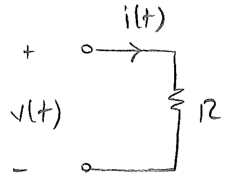
Instantaneous & average power in SSS

Resistor



Let $i(t) = I_m \cos(wt + \phi)$, period $T = \frac{2\pi}{w}$ sec , $I_{eff} = \frac{I_m}{\sqrt{2}}$

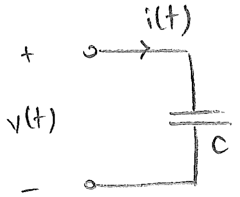
$$\Rightarrow p(t) = i(t)v(t)$$
$$= R I_m \cos(wt+\phi) I_m \cos(wt+\phi)$$
$$= \frac{1}{2} R I_m^2 \left(1 + \cos[2(wt+\phi)]\right) \qquad \cos^2\alpha = \frac{1}{2}[1+\cos 2\alpha]$$

Hence inst. power : $p(t) = \frac{1}{2} R I_m^2 \left(1 + \cos(2wt + 2\phi)\right)$, periodic with T

aver. power : $P_{av} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{2} R I_m^2 \left\{ \underbrace{\frac{1}{T}\int_0^T dt}_{=1} + \underbrace{\frac{1}{T}\int_0^T \cos(2wt+2\phi)dt}_{=0} \right\}$

$$\Rightarrow P_{av} = \frac{1}{2} R I_m^2 = \boxed{R I_{eff}^2}$$

Capacitor



Let $v(t) = V_m \cos(wt + \phi)$, period $T = \frac{2\pi}{w}$ sec , $V_{eff} = \frac{V_m}{\sqrt{2}}$

$$\Rightarrow I = jwC V = jwC V_m \angle\phi = wC V_m \angle \phi + \frac{\pi}{2}$$

$$\Rightarrow i(t) = wC V_m \cos(wt + \phi + \frac{\pi}{2})$$

$$\Rightarrow p(t) = i(t)v(t) = wC V_m^2 \cos(wt+\phi)\cos(wt+\phi+\frac{\pi}{2})$$
$$= \frac{1}{2} wC V_m^2 \cos(2wt + 2\phi + \frac{\pi}{2}) \qquad \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

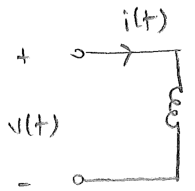
$\Rightarrow \boxed{P_{av} = 0}$ because $p(t)$ is purely sinusoidal.

How about stored energy?

$e(t) = \frac{1}{2} C v(t)^2 = \frac{1}{2} C V_m^2 \cos^2(wt+\phi)$, periodic with T

$$\Rightarrow E_{av} = \frac{1}{T}\int_0^T e(t)dt = \frac{1}{4} C V_m^2 = \boxed{\frac{1}{2} C V_{eff}^2}$$

Inductor



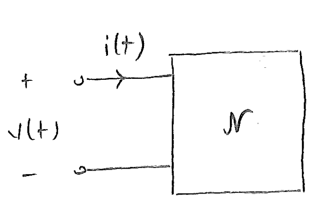
Let $i(t) = I_m \cos(\omega t + \phi)$, period $T = \frac{2\pi}{\omega}$, $I_{eff} = \frac{I_m}{\sqrt{2}}$

By duality we can write

$p(t) = \frac{1}{2} \omega L I_m^2 \cos(2\omega t + 2\phi + \frac{\pi}{2}) \Rightarrow \boxed{P_{av} = 0}$

$e(t) = \frac{1}{2} L I_m^2 \cos^2(\omega t + \phi) \Rightarrow E_{av} = \frac{1}{4} L I_m^2 = \boxed{\frac{1}{2} L I_{eff}^2}$

Power into one-port



Let $i(t) = I_m \cos(\omega t + \phi_i)$

$v(t) = V_m \cos(\omega t + \phi_v)$

$$\Rightarrow p(t) = i(t) v(t)$$

$$= I_m V_m \cos(\omega t + \phi_i) \cos(\omega t + \phi_v)$$

$$= \frac{1}{2} I_m V_m \cos(\phi_v - \phi_i) + \underbrace{\frac{1}{2} I_m V_m \cos(2\omega t + \phi_i + \phi_v)}_{\text{purely sinusoidal term, i.e., has zero average}}$$

$$\Rightarrow P_{av} = \frac{1}{2} I_m V_m \cos(\phi_v - \phi_i) = \boxed{I_{eff} V_{eff} \cos(\phi_v - \phi_i)}$$

The term "$\cos(\phi_v - \phi_i)$" is called the <u>power factor</u> when $\mathcal{N}$ is passive and ind. source free.

Now, suppose $\mathcal{N}$ contains no ind. sources and therefore can be represented by an impedance Z. That is, $V = Z I$.

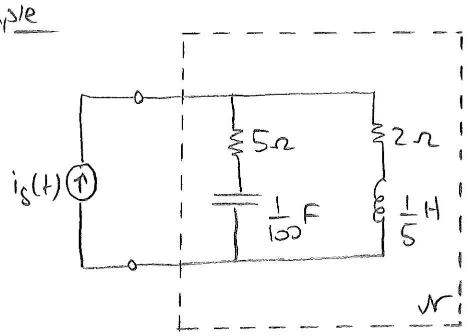
Observe: $Z = \frac{V_m \angle \phi_v}{I_m \angle \phi_i} = \frac{V_m}{I_m} \angle \phi_v - \phi_i = |Z| \angle \phi_v - \phi_i$

Then $P_{av} = \frac{1}{2} V_m I_m \cos(\phi_v - \phi_i) = \frac{1}{2} I_m \{ I_m |Z| \} \cos(\phi_v - \phi_i)$

$$= \frac{1}{2} I_m^2 \{ |Z| \cos(\phi_v - \phi_i) \}$$

Hence $P_{av} = \frac{1}{2} I_m^2 \, \mathrm{Re}\{Z\} = \boxed{I_{eff}^2 \cdot \mathrm{Re}\{Z\}}$

Example



$i_s(t) = 5\cos(20t)$ A

The circuit is in SSS. Find:

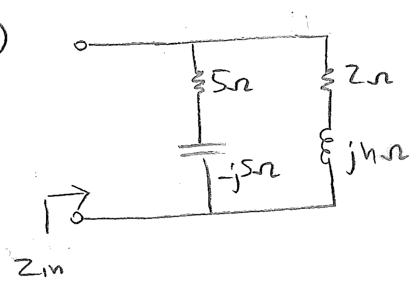
a) The average power delivered to $\mathcal{N}$.

b) The aver. power dissipated on 5Ω resistor.

c) The aver. power dissipated on 2Ω resistor.

d) The aver. stored energy in the inductor.

Sol'n a)



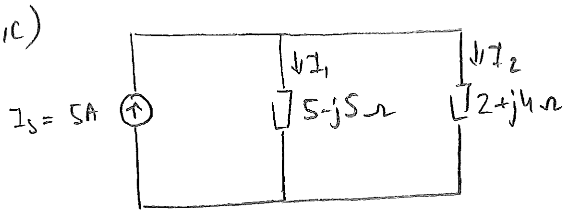
$$\frac{1}{Z_{in}} = \frac{1}{5-j5} + \frac{1}{2+j4} = \frac{5+j5}{50} + \frac{2-j4}{20}$$

$$= \frac{1+j}{10} + \frac{1-j2}{10} = \frac{2-j}{10}$$

$$\Rightarrow Z_{in} = \frac{10}{2-j} = \frac{\cancel{10}^{2}(2+j)}{\cancel{5}} = 4+j2\ \Omega$$

$$\Rightarrow P_{\mathcal{N}} = I_{s,eff}^2 \cdot Re\{Z_{in}\} = \left(\frac{5}{\sqrt{2}}\right)^2 \cdot 4 = \boxed{50W}$$

b,c)



$$I_1 = \frac{Z_2}{Z_1+Z_2} I_s = \frac{2+j4}{7-j} \cdot 5 = 1+j3 \text{ A}$$

$$\Rightarrow I_2 = I_s - I_1 = 4-j3 \text{ A}$$

$$\Rightarrow I_1 = \sqrt{10} \angle \tan^{-1} 3 \text{ A} \quad \& \quad I_2 = 5 \angle -\tan^{-1} 3/4 \text{ A}$$

$$\Rightarrow P_{5\Omega} = I_{1,eff}^2 \cdot 5 = \left(\frac{\sqrt{10}}{\sqrt{2}}\right)^2 \cdot 5 = \boxed{25W} \quad \& \quad P_{2\Omega} = I_{2,eff}^2 \cdot 2 = \boxed{25W}$$

Note that $P_{\mathcal{N}} = P_{5\Omega} + P_{2\Omega}$!

d) $$E_L = \frac{1}{2} \cdot \frac{1}{5} \cdot I_{2,eff}^2 = \frac{1}{10} \cdot \left(\frac{5}{\sqrt{2}}\right)^2 = \frac{25}{20} = \boxed{\frac{5}{4} J}$$