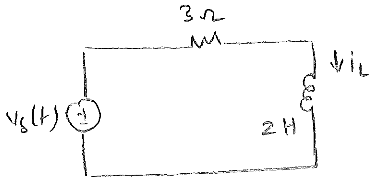
Ch. V — Sinusoidal Steady State Analysis

Example (Revisited)



$v_s(t) = 10\cos 2t$ V

$i_L(0) = 0$

Diff. eqn. $Di_L + \frac{3}{2} i_L = 5\cos 2t$, Nat. freq. $s = -\frac{3}{2}$

Sol'n $i_L(t) = \underbrace{-\frac{6}{5}e^{-\frac{3}{2}t}}_{\substack{\text{transient}\\ \text{part}\\ \text{(hmg. sol'n)}}} + \underbrace{\frac{6}{5}\cos 2t + \frac{8}{5}\sin 2t}_{\substack{\text{sinusoidal steady state}\\ \text{part}\\ \text{(particular sol'n)}}}$ A

Question : Can we compute SSS solution without solving the diff. eqn?

Answer : YES!

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Consider an LTI circuit such that

→ Each natural freq. $\lambda_i = \alpha_i + j\omega_i$ satisfies $\mathrm{Re}\{\lambda_i\} = \alpha_i < 0$. This implies that the hmg. sol'n always decays to zero, regardless of the init. cond.

→ The inputs are sinusoidal signals at the same frequency ω rad/sec.

In such a circuit the response converges to the particular solution which is also a sinusoid with frequency ω (the freq. of the inputs). This solution is called the sinusoidal steady state (SSS) solution.

Some useful identities

1) $e^{j\phi} = \cos\phi + j\sin\phi$ (Euler's identity)

2) $\mathrm{Re}\{z_1 + z_2\} = \mathrm{Re}\{z_1\} + \mathrm{Re}\{z_2\}$

3) $D\,\mathrm{Re}\{f(t)\} = \mathrm{Re}\{Df(t)\}$

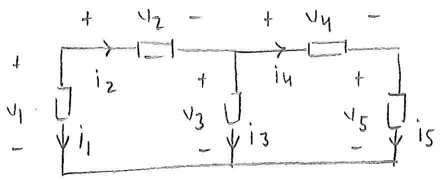
4) Let $\omega \neq 0$ be real. If $\mathrm{Re}\{X_1 e^{j\omega t}\} = \mathrm{Re}\{X_2 e^{j\omega t}\}$ for all $t \in \mathbb{R}$ then $X_1 = X_2$.

proof $t = 0 \Rightarrow \mathrm{Re}\{X_1\} = \mathrm{Re}\{X_2\}$ (*)

$t = \frac{3\pi}{2\omega} \Rightarrow \mathrm{Re}\{-jX_1\} = \mathrm{Re}\{-jX_2\} \Rightarrow \mathrm{Im}\{X_1\} = \mathrm{Im}\{X_2\}$ (**)

(*) & (**) $\Rightarrow X_1 = X_2$

Consider a circuit in sinusoidal steady state



(input frequency : ω rad/sec)

Since the circuit is in SSS, each voltage/current signal should look like

$x_k(t) = A_k \cos(\omega t + \phi_k)$ for some magnitude A_k and phase ϕ_k.

Definition For a signal $x_k(t) = A_k \cos(\omega t + \phi_k)$, the complex number

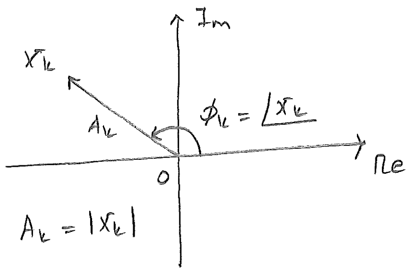
$X_k := A_k e^{j\phi_k}$ is called the phasor of $x_k(t)$.

Note that given the pair (X_k, ω) we can reconstruct the signal $x_k(t)$ as

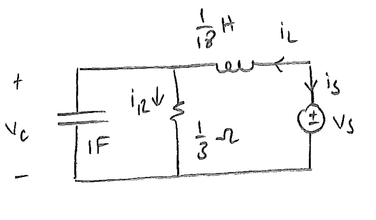
phasor of $x_k(t)$; the operating freq. of the circuit

$$x_k(t) = \mathrm{Re}\{X_k e^{j\omega t}\} = A_k \cos(\omega t + \phi_k)$$

Phasor diagram contains the vector representation of X_k on the complex plane :



Example [circuit in SSS]



$v_s(t) = \cos 3t$ V

Signals

$i_L(t) = 6\cos 3t$ A
$i_R(t) = 3\sqrt{2}\cos(3t - \frac{\pi}{4})$ A
$i_c(t) = 3\sqrt{2}\cos(3t + \frac{\pi}{4})$ A
$i_s(t) = -6\cos 3t$ A

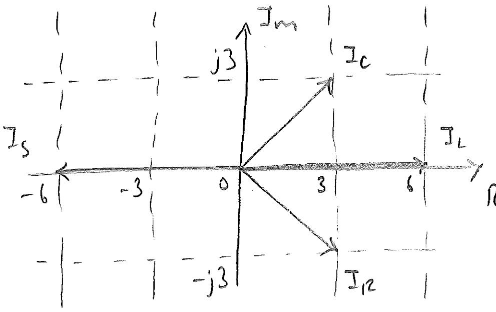
$v_L(t) = -\sin 3t = \cos(3t + \frac{\pi}{2})$ V
$v_R(t) = \sqrt{2}\cos(3t - \frac{\pi}{4})$ V
$v_c(t) = \sqrt{2}\cos(3t - \frac{\pi}{4})$ V
$v_s(t) = \cos 3t$ V

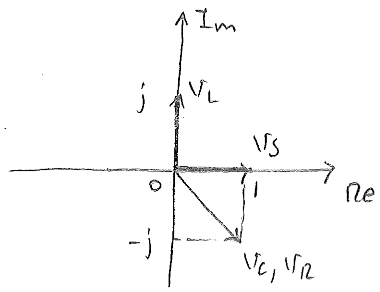
Phasors

$I_L = 6e^{j0} = 6$ A
$I_R = 3\sqrt{2}e^{-j\frac{\pi}{4}} = 3 - j3$ A
$I_c = 3\sqrt{2}e^{j\frac{\pi}{4}} = 3 + j3$ A
$I_s = 6e^{j\pi} = -6$ A

$V_L = e^{j\frac{\pi}{2}} = j$ V
$V_R = \sqrt{2}e^{-j\frac{\pi}{4}} = 1 - j$ V
$V_c = \sqrt{2}e^{-j\frac{\pi}{4}} = 1 - j$ V
$V_s = e^{j0} = 1$ V

Phasor Diagram





Note : $I_L = I_C + I_R$ (KCL ?)

Note: $V_S = V_C + V_L$ (KVL ?)

Claim Phasors of currents obey KCL.

Proof Let some currents satisfy KCL: $i_1(t) + i_2(t) + \cdots + i_n(t) = 0$

$$\Rightarrow \operatorname{Re}\{I_1 e^{j\omega t}\} + \operatorname{Re}\{I_2 e^{j\omega t}\} + \cdots + \operatorname{Re}\{I_n e^{j\omega t}\} = 0$$

$$\Rightarrow \operatorname{Re}\{I_1 e^{j\omega t} + I_2 e^{j\omega t} + \cdots + I_n e^{j\omega t}\} = 0$$

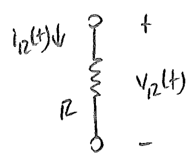
$$\Rightarrow \operatorname{Re}\{(I_1 + I_2 + \cdots + I_n) e^{j\omega t}\} = 0$$

$$\Rightarrow I_1 + I_2 + \cdots + I_n = 0 \quad \blacksquare$$

Likewise, phasors of voltages obey KVL.

Terminal Equations in Phasors

Resistor



(belongs to a circuit in SSS at freq. ω rad/sec)

$v_R(t) = \operatorname{Re}\{V_R e^{j\omega t}\}$, V_R: phasor of $v_R(t)$

$i_R(t) = \operatorname{Re}\{I_R e^{j\omega t}\}$, I_R : phasor of $i_R(t)$

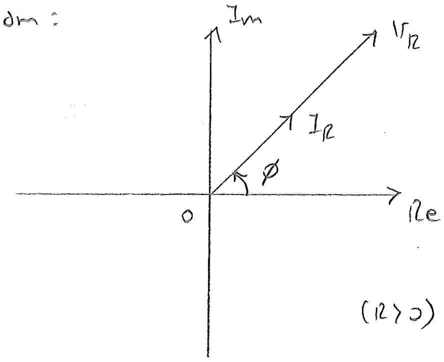
$V_R \sim I_R$?

$v_R(t) = R\, i_R(t)$

$$\Rightarrow \operatorname{Re}\{V_R e^{j\omega t}\} = R \cdot \operatorname{Re}\{I_R e^{j\omega t}\}$$

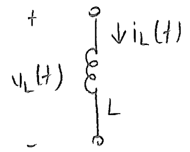
$$= \operatorname{Re}\{R I_R e^{j\omega t}\} \Rightarrow \boxed{V_R = R\, I_R}$$

phasor diagram:



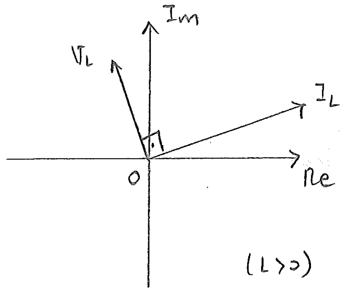
Both V_R & I_R have the same phase (ϕ). Such signals are said to be <u>in phase</u>

<u>Inductor</u>



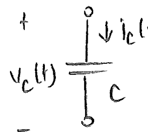
$$v_L(t) = Re\{V_L e^{j\omega t}\}$$
$$i_L(t) = Re\{I_L e^{j\omega t}\}$$

$$v_L(t) = L\,D\,i_L(t) \Rightarrow Re\{V_L e^{j\omega t}\} = L\,D\,Re\{I_L e^{j\omega t}\}$$
$$= Re\{L I_L D e^{j\omega t}\}$$
$$= Re\{j\omega L I_L e^{j\omega t}\} \Rightarrow \boxed{V_L = j\omega L I_L}$$



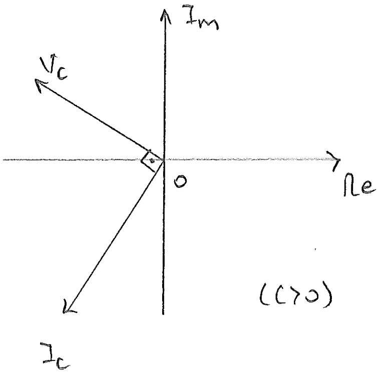
The current phasor I_L <u>lags</u> the voltage phasor V_L by 90°. (or voltage <u>leads</u> current by 90°.)

<u>Capacitor</u>



$$v_c(t) = Re\{V_c e^{j\omega t}\}$$
$$i_c(t) = Re\{I_c e^{j\omega t}\}$$

Dual of inductor $\Rightarrow$ $\boxed{I_c = j\omega C V_c}$



The current leads the voltage by 90°.
(or, voltage lags current by 90°.)

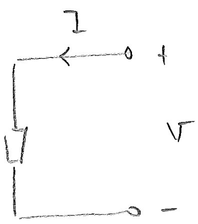
Impedance & Admittance

Consider a one-port that

→ is part of a circuit in SSS

→ contains no independent sources

→ contains the control variables of its dependent sources



(I, V : phasors)

we define :

Impedance $Z := \frac{V}{I} = |Z| e^{j\theta_Z} = |Z| \angle \theta_Z$ (measured in ohms)

admittance $Y := \frac{I}{V} = \frac{1}{Z}$ (measured in mhos ℧)

Remark Impedance is the generalization of resistance in phasor domain. It is a function of the operating frequency ω.

Impedance Z in rectangular form : $Z = R + jX$

$R = \mathrm{Re}\{Z\}$: resistance

$X = \mathrm{Im}\{Z\}$: reactance

Likewise, $Y = G + jB$

$G = \mathrm{Re}\{Y\}$: conductance

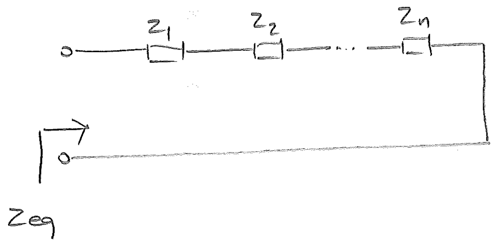
$B = \mathrm{Im}\{Y\}$: susceptance

Note : $R \neq \frac{1}{G}$ & $X \neq \frac{1}{B}$ in general !

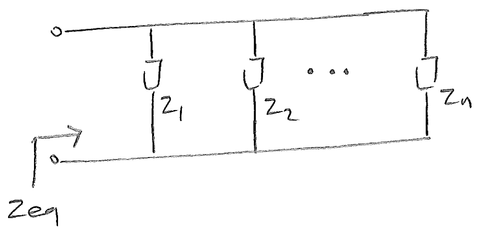
We have

Component	Impedance	Admittance
R	$Z_R = R$	$Y_R = \frac{1}{R}$
L	$Z_L = j\omega L$	$Y_L = \frac{1}{j\omega L}$
C	$Z_C = \frac{1}{j\omega C}$	$Y_C = j\omega C$

Series & parallel connections

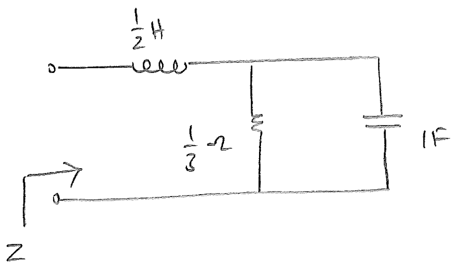


$$Z_{eq} = Z_1 + Z_2 + \dots + Z_n$$



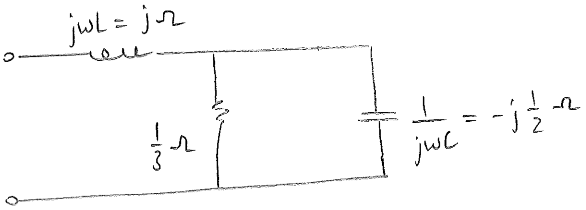
$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \dots + \frac{1}{Z_n}$$

Example

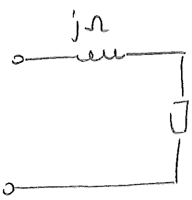


Find Z for $w = 2$ rad/sec

Sol'n



$\equiv$

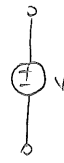


$$\frac{1}{3} \,//\, -j\frac{1}{2} = \frac{1}{3+j2} = \frac{3}{13} - j\frac{2}{13}\,\Omega$$

$\equiv$

$$j \,//\, \frac{3}{13} - j\frac{2}{13} = \boxed{\frac{3}{13} + j\frac{11}{13}\,\Omega}$$

Independent Sources (in SSS)



$v_s(t) = V_m \cos(\omega t + \phi_v)$ $\xrightarrow{\text{in phasor domain}}$

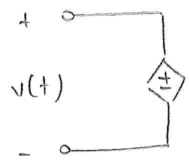
$V_s = V_m \angle \phi_v$

phasor of $v_s(t)$

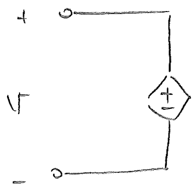
$i_s(t) = I_m \cos(\omega t + \phi_i)$ $\xrightarrow{\text{P.D.}}$

$I_s = I_m \angle \phi_i$

Dependent Sources



$v(t) = \mu i_x(t)$ $\xrightarrow{\text{P.D.}}$



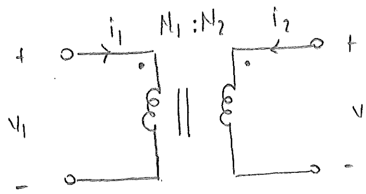
$V = \mu I_x$

phasor of $v(t)$

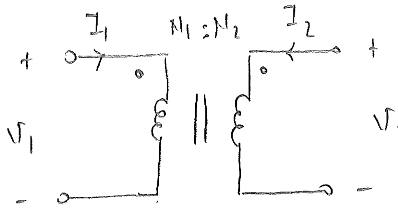
phasor of $I_x(t)$

i_x: control current

Ideal transformer



$\xrightarrow{\text{P.D.}}$



$\frac{v_1}{N_1} = \frac{v_2}{N_2}$ & $N_1 i_1 + N_2 i_2 = 0$

$\frac{V_1}{N_1} = \frac{V_2}{N_2}$ & $N_1 I_1 + N_2 I_2 = 0$