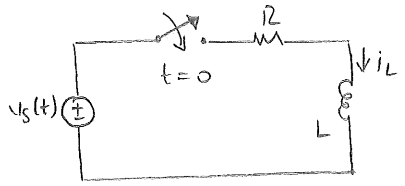
Example [sinusoidal input]



Find $i_L(t)$ for $t>0$.

$v_s(t) = V_0 \cos(\omega t + \phi)$

Sol'n $\quad L D i_L = v_L = v_s - R i_L \quad \Rightarrow \quad D i_L + \frac{R}{L} i_L = \frac{1}{L} v_s$

Hence $\quad D i_L + \frac{R}{L} i_L = \frac{V_0}{L}\cos(\omega t + \phi)$ [Diff. eqn.] $\quad$ (Note: natural freq. $s = -\frac{R}{L}$)

$i_L(0) = 0$ [Init. cond.]

Then hmg. sol'n : $i_h(t) = K e^{-\frac{R}{L}t}$

part. sol'n : $i_p(t) = A\cos\omega t + B\sin\omega t$ (i.e. a sinusoidal function whose freq. ω same as that of the input.)

To be determined : A, B, K (use the diff. eqn. to find A, B; use the initial cond. constraint to find K)

$D i_p + \frac{R}{L} i_p = \frac{V_0}{L}\cos(\omega t + \phi) \qquad \{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta\}$

$$\Rightarrow \frac{d}{dt}\{A\cos\omega t + B\sin\omega t\} + \frac{R}{L}\{A\cos\omega t + B\sin\omega t\} = \frac{V_0}{L}\{\cos\phi\cos\omega t - \sin\phi\sin\omega t\}$$

$$\Rightarrow -A\omega\sin\omega t + B\omega\cos\omega t + \frac{AR}{L}\cos\omega t + \frac{BR}{L}\sin\omega t = \frac{V_0\cos\phi}{L}\cos\omega t - \frac{V_0\sin\phi}{L}\sin\omega t$$

$$\Rightarrow \underbrace{\left\{B\omega + \frac{AR}{L} - \frac{V_0\cos\phi}{L}\right\}}_{=0}\cos\omega t + \underbrace{\left\{-A\omega + \frac{BR}{L} + \frac{V_0\sin\phi}{L}\right\}}_{=0}\sin\omega t = 0$$

$$\Rightarrow \begin{bmatrix} \frac{R}{L} & \omega \\ -\omega & \frac{R}{L} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{V_0}{L}\cos\phi \\ -\frac{V_0}{L}\sin\phi \end{bmatrix} \Rightarrow \begin{bmatrix} A \\ B \end{bmatrix} = \frac{V_0/L}{\frac{R^2}{L^2} + \omega^2}\begin{bmatrix} \frac{R}{L} & -\omega \\ \omega & \frac{R}{L} \end{bmatrix}\begin{bmatrix} \cos\phi \\ -\sin\phi \end{bmatrix}$$

$\Rightarrow i_L(t) = i_h(t) + i_p(t) = K e^{-\frac{R}{L}t} + A\cos wt + B\sin wt$

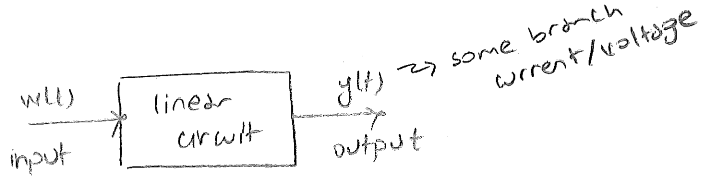
$i_L(0) = 0 \Rightarrow K = -A$

Hence $\boxed{i_L(t) = -A e^{-\frac{R}{L}t} + A\cos wt + B\sin wt}$ $(t>0)$

Let $R = 3\Omega$, $L = 2H$, $w = 2\,rad/sec$, $\phi = 0°$, $V_0 = 10V$

$\Rightarrow A = \frac{6}{5}$ & $B = \frac{8}{5}$ $\Rightarrow$ $i_L(t) = \underbrace{-\frac{6}{5}e^{-\frac{3}{2}t}}_{\text{transient part}} + \underbrace{\frac{6}{5}\cos 2t + \frac{8}{5}\sin 2t}_{\text{(sinusoidal) steady state part}}$ Amps

Superposition in linear dynamic circuits



Note that

$y(t) = y(t, x, w(\cdot))$

x: initial condition (vector)

$w(\cdot)$: input signal (vector)

init. cond. $x = [v_{C_1}(0^-) \cdots v_{C_m}(0^-), i_{L_1}(0^-) \cdots i_{L_n}(0^-)]^T$

input $w(t) = [v_{s_1}(t) \cdots v_{s_p}(t), i_{s_1}(t) \cdots i_{s_q}(t)]^T$

Remark For a first-order circuit $x = v_C(0^-)$ or $x = i_L(0^-)$.

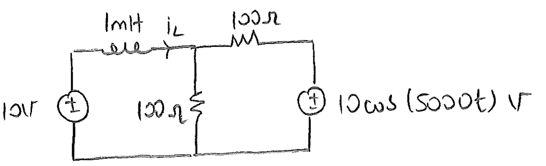
We have:

1) $y(t, \alpha_1 x_1 + \alpha_2 x_2, 0) = \alpha_1 y(t, x_1, 0) + \alpha_2 y(t, x_2, 0)$ superposition w.r.t. init. cond.

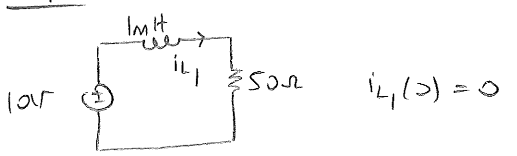
2) $y(t, 0, \beta_1 w_1 + \beta_2 w_2) = \beta_1 y(t, 0, w_1) + \beta_2 y(t, 0, w_2)$ superposition w.r.t. inputs

3) $y(t, \alpha x, \beta w) = \alpha y(t, x, 0) + \beta y(t, 0, w)$ superposition w.r.t. init. cond. & input

Exercise Find the zero-state response i_L

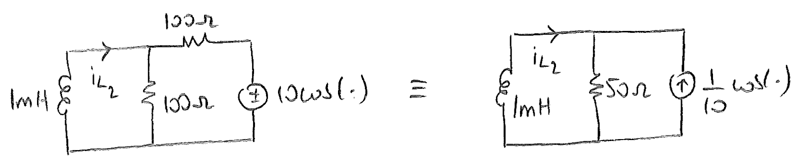


Step 1 Kill AC input to find i_{L_1}



$i_{L_1}(0) = 0$

Step 2 Kill DC input to find i_{L_2}



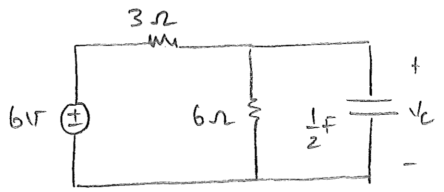
$i_{L_2}(0) = 0$

Step 3 Superpose

$i_L(t) = i_{L_1}(t) + i_{L_2}(t)$

Complete response [nonzero input + nonzero init. cond.]

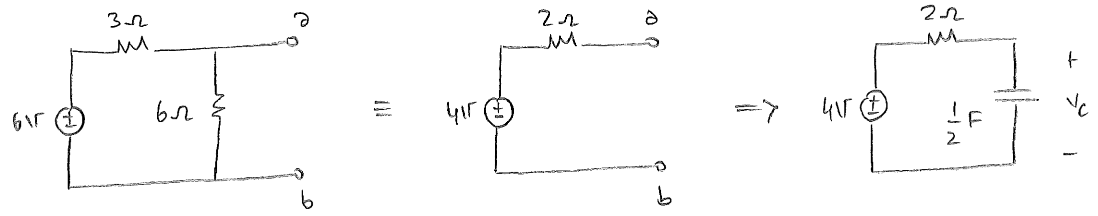
Example



$v_c(0) = 2V$

$v_c(t) = ?$ for $t \geq 0$

Sol'n



$v_c(0) = 2V$

KVL : $0 = -4 + 2i_c + v_c = -4 + 2\{\frac{1}{2}Dv_c\} + v_c \Rightarrow Dv_c + v_c = 4$ (diff. eqn.)

hmg sol'n : $v_h(t) = Ke^{-t}$

part. sol'n : $v_p(t) = 4$

$\Big\}\ v_c(t) = 4 + Ke^{-t} \xrightarrow[(v_c(0)=2V)]{} \boxed{v_c(t) = 4 - 2e^{-t}\ V}$

Another approach (superposition)

Step 1 Find zero-state resp. $v_{zs}(t)$

$Dv_c + v_c = 4$ with $v_c(0) = 0 \Rightarrow v_{zs}(t) = 4 - 4e^{-t}\ V$

Step 2 Find zero-input resp. $v_{zi}(t)$

$Dv_c + v_c = 0$ with $v_c(0) = 2V \Rightarrow v_{zi}(t) = 2e^{-t}\ V$

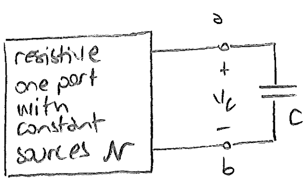
Step 3 Superpose

$v_c(t) = v_{zs}(t) + v_{zi}(t)$

$= 4 - 2e^{-t}\ V$

(as expected)

Yet another approach



$v_c(0) = V_0$

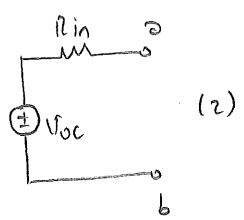
what do we know?

→ form of diff eqn. $Dv_c + \frac{1}{\tau}v_c = \text{constant}$

→ form of sol'n $v_c(t) = A + Be^{-t/\tau}$ (1)

How to find A, B, τ ?

Let $N \equiv$



(2)

(1) => $v_c(\infty) = A$

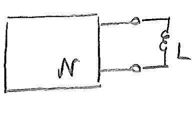
(2) => $v_c(\infty) = V_{oc}$

$\Big\}\ \boxed{A = V_{oc}}$

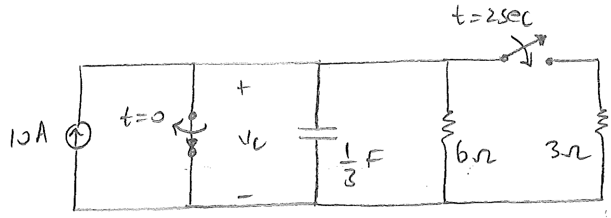
(2) => $\boxed{\tau = R_{in}C}$

(1) => $A + B = v_c(0) \Rightarrow B = v_c(0) - v_c(\infty) \Rightarrow \boxed{B = V_0 - V_{oc}}$

Exercise Work out the dual case

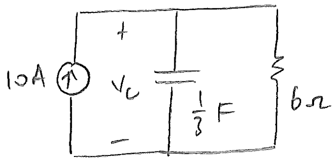


Example



Find $v_c(t)$ for $t \geq 0$.

Sol'n $0 \leq t < 2$



$v_c(0) = 0$

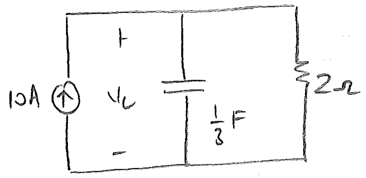
KCL $\Rightarrow \frac{1}{3} D v_c + \frac{v_c}{6} = 10 \Rightarrow D v_c + \frac{1}{2} v_c = 30$

$$\left.\begin{aligned} v_h(t) &= k e^{-t/2} \\ v_p(t) &= 60\,V \end{aligned}\right\} v_c(t) = 60 + k e^{-t/2} \quad \leftarrow v_c(0) = 0$$

$$= 60 - 60 e^{-t/2}\ V$$

at $t = 2$ $\quad v_c(2) = 60\left(1 - \frac{1}{e}\right) =: V_2 \cong 38V$

$t \geq 2$



$v_c(2) = V_2$

$D v_c + \frac{3}{2} v_c = 30$

$$\left.\begin{aligned} v_h(t) &= k e^{-\frac{3}{2}(t-2)} \\ v_p(t) &= 20V \end{aligned}\right\} v_c(t) = 20 + k e^{-\frac{3}{2}(t-2)} \quad \leftarrow v_c(2) = V_2$$

$$= 20 + [V_2 - 20] e^{-\frac{3}{2}(t-2)}\ V$$

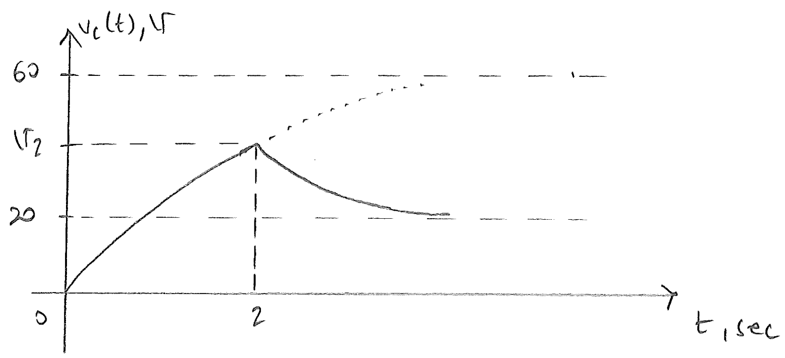
or

$$\left.\begin{aligned} v_h(t) &= \tilde{k} e^{-\frac{3}{2}t} \\ v_p(t) &= 20V \end{aligned}\right\} v_c(t) = 20 + \tilde{k} e^{-\frac{3}{2}t} \quad \leftarrow v_c(2) = V_2$$

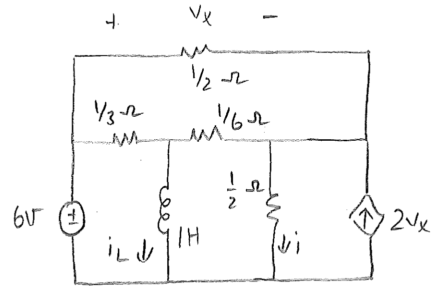
$$= 20 + \left[(V_2 - 20) e^{3}\right] e^{-\frac{3}{2}t}\ V$$

Hence

$$v_c(t) = \begin{cases} 60\left[1 - e^{-t/2}\right]\ V & \text{for } 0 \leq t < 2 \\ 20 + [V_2 - 20] e^{-\frac{3}{2}(t-2)}\ V & \text{for } t \geq 2 \end{cases}$$

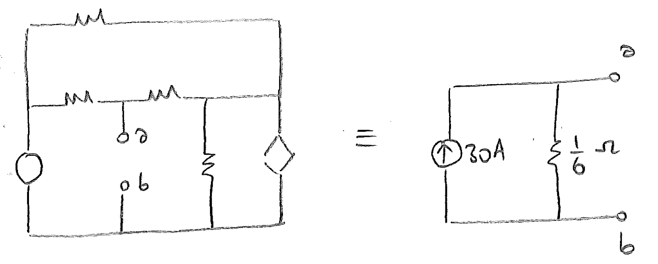


Example

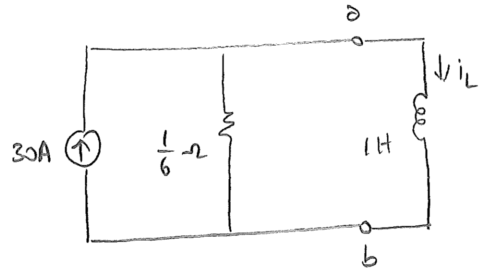


$i_L(0) = 6A$, $i(t) = ?$ for $t \geq 0$

Step 1 Find the Norton / Thevenin equiv. circuit seen by the inductor



Step 2 Using the equiv. circuit find $i_L(t)$



KCL: $-30 + i_L + 6 L D i_L = 0$

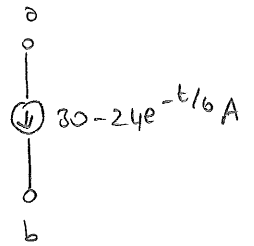
$\Rightarrow D i_L + \frac{1}{6} i_L = 5 \quad \Rightarrow i_L(t) = 30 + K e^{-t/6}$

$i_L(0) = 6A \Rightarrow K = -24 \quad \Rightarrow i_L(t) = 30 - 24 e^{-t/6}$ A

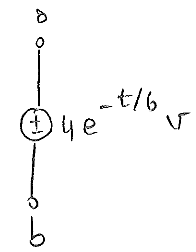
$\Rightarrow v_L(t) = 4 e^{-t/6}$ V

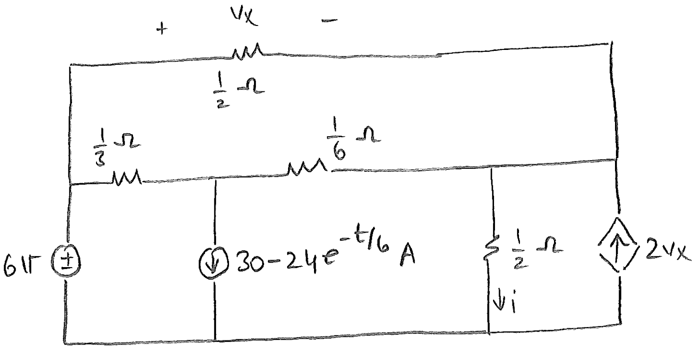
Step 3 Substitute

with



or



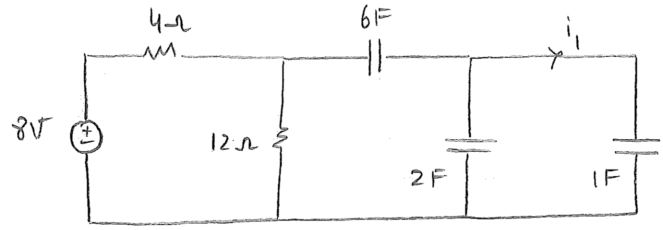


Solve for i

Answer : $\boxed{i(t) = 4 + 4e^{-t/6} \text{ A}}$

for $t \geq 0$

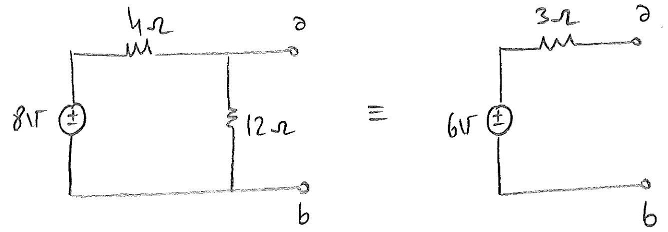
Example



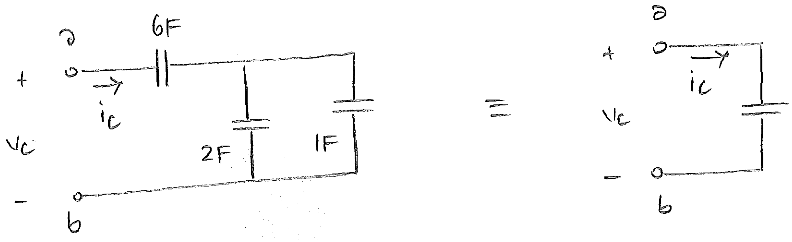
Capacitors initially uncharged.

Find $i_1(t)$ for $t \geq 0$.

Sol'n



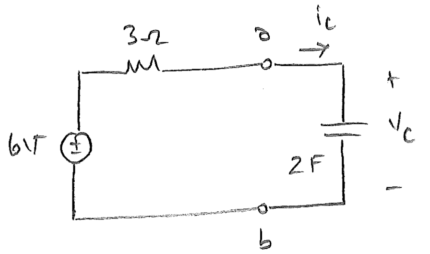
and



$2F = (2 /\!/ 1) \gg 6$

$v_c(0) = 0$

Hence



$v_c(0) = 0$

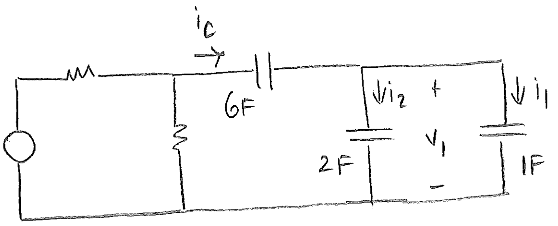
$3(2Dv_c) + v_c = 6 \quad \Rightarrow \quad Dv_c + \frac{1}{6} v_c = 1$

$\Rightarrow v_c(t) = 6 + K e^{-t/6}$ $v_c(0) = 0$

$= 6 - 6e^{-t/6}$ V

$\Rightarrow i_c(t) = 2Dv_c = 2e^{-t/6}$ A

Return to the original configuration



$\left.\begin{array}{l} i_1 = Dv_1 \\ i_2 = 2Dv_1 \end{array}\right\} \Rightarrow i_2 = 2i_1$

$i_c = i_1 + i_2 = 3i_1 \quad \Rightarrow \quad i_1 = \frac{1}{3} i_c$

$\Rightarrow \boxed{i_1(t) = \frac{2}{3} e^{-t/6} \text{ A}}$ for $t \geq 0$