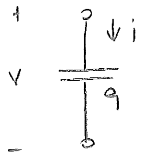
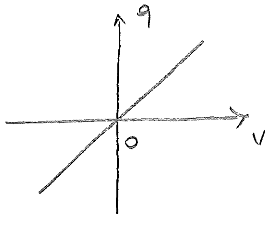
Ch. IV

First-Order Circuits

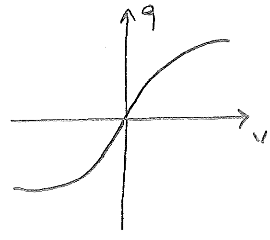
Capacitor A 2-terminal element that can store charge. The charge q it stores and the voltage v across its terminals satisfy a relation described by a curve on the q-v plane.



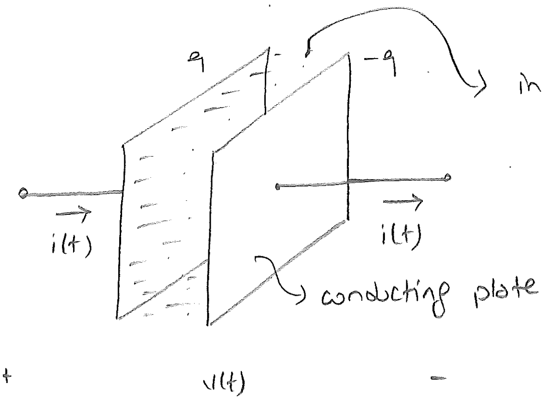
Ex



linear capacitor



nonlinear cap.



$$i(t) = \frac{d}{dt} q(t) \quad \text{(def. of current)}$$

$$\Rightarrow \quad q(t) = q(t_0) + \int_{t_0}^{t} i(\tau)\,d\tau$$

LTI Capacitor

$\boxed{q(t) = C\,v(t)}$ C: constant named capacitance, measured in Farads (F)

i-v relation?

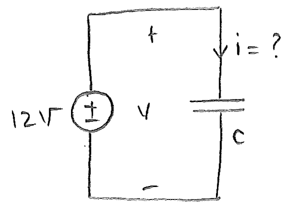
$$i(t) = \frac{d}{dt} q(t) = \frac{d}{dt}\{C\,v(t)\} = C\,\frac{dv(t)}{dt} \quad \Rightarrow \quad \boxed{i(t) = C\,\frac{dv(t)}{dt}}$$

Equivalently,

$$v(t) = \frac{1}{C} q(t) = \frac{1}{C}\left\{ q(t_0) + \int_{t_0}^{t} i(\tau)\,d\tau \right\} \quad \Rightarrow \quad \boxed{v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau}$$

($v(t_0)$: initial voltage of the capacitor.)

Ex



$$i = C\,\cancel{\frac{dv}{dt}}^{\,0} = 0$$

Therefore the capacitor behaves as open circuit under DC (constant) voltage.

#2 18 Kasım 2023

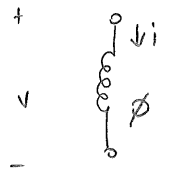
power $p(t) = i(t)v(t)$

$= C\dfrac{dv(t)}{dt}v(t)$ $\Bigg\}$ $p(t) = Cv(t)\dfrac{dv(t)}{dt}$

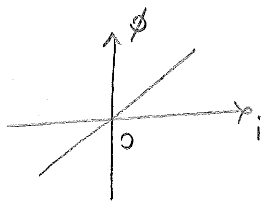
stored energy $(C>0)$ $E(t) = \int_{-\infty}^{t} p(\tau)d\tau = \int_{\underbrace{v(-\infty)}_{=0}}^{v(t)} Cv\,dv = \frac{1}{2}Cv(t)^2$

Hence, the energy stored in an LTI capacitor : $\boxed{E = \frac{1}{2}Cv^2}$

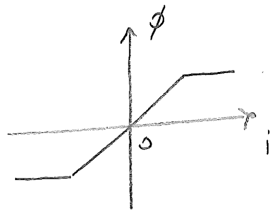
Inductor A 2-terminal element whose magnetic flux ϕ and current i satisfy a relation described by a curve on the ϕ-i plane.



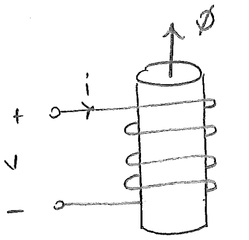
Ex



linear inductor



nonlinear inductor



Faraday's Law : $v(t) = \dfrac{d}{dt}\phi(t)$

LTI inductor

$\boxed{\phi(t) = Li(t)}$ L: inductance, measured in Henries (H)

i-v relation ? $v(t) = \dfrac{d}{dt}\phi(t) = \dfrac{d}{dt}\{Li(t)\} = L\dfrac{di(t)}{dt} \Rightarrow \boxed{v(t) = L\dfrac{di(t)}{dt}}$

Equivalently, $\boxed{i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau}$

power
$$p(t) = v(t)i(t) = L\frac{di(t)}{dt}i(t) \Bigg\} \quad p(t) = L\,i(t)\frac{di(t)}{dt}$$

Stored energy $(L>0)$
$$E(t) = \int_{-\infty}^{t} p(\tau)d\tau = \int_{\underbrace{i(-\infty)}_{=0}}^{i(t)} L\,i\,di = \frac{1}{2}L\,i(t)^2$$

$$\Rightarrow \boxed{E = \frac{1}{2}Li^2}$$

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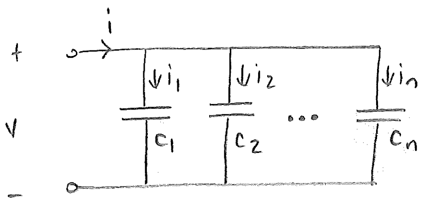
Capacitors in parallel connection



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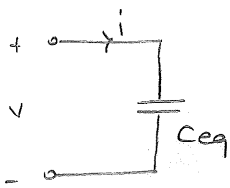
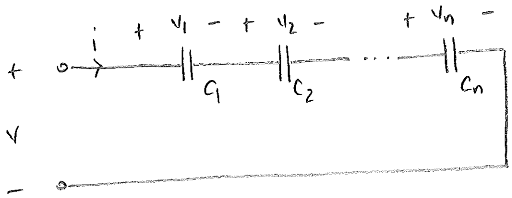


$$i(t) = \underbrace{C_{eq}}_{?}\frac{dv(t)}{dt}$$

$v(t_0) = V_0$ $\qquad$ $v(t_0) = V_0$

$$i(t) = i_1(t) + i_2(t) + \cdots + i_n(t) = C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + \cdots + C_n\frac{dv}{dt}$$
$$= \underbrace{[C_1 + C_2 + \cdots + C_n]}_{C_{eq}}\frac{dv}{dt} \quad \Rightarrow \boxed{C_{eq} = C_1 + C_2 + \cdots + C_n}$$

Capacitors in series connection



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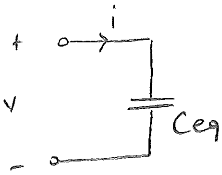


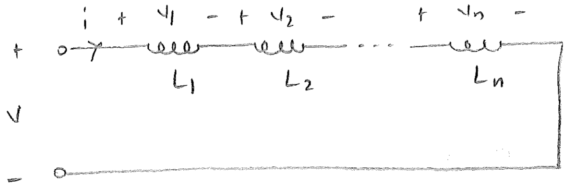
$$v(t) = v(t_0) + \underbrace{\frac{1}{C_{eq}}}_{?}\int_{t_0}^{t} i(\tau)d\tau$$

$v_1(t_0) = V_{10}$, $v_2(t_0) = V_{20}$, ..., $v_n(t_0) = V_{n0}$ $\qquad$ $v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_n(t_0)$

$$v(t) = v_1(t) + v_2(t) + \cdots + v_n(t) = \left\{v_1(t_0) + \frac{1}{C_1}\int_{t_0}^{t} i(\tau)d\tau\right\} + \cdots + \left\{v_n(t_0) + \frac{1}{C_n}\int_{t_0}^{t} i(\tau)d\tau\right\}$$
$$= \underbrace{\left\{V_{10} + V_{20} + \cdots + V_{n0}\right\}}_{v(t_0)} + \underbrace{\left[\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}\right]}_{\frac{1}{C_{eq}}}\int_{t_0}^{t} i(\tau)d\tau$$

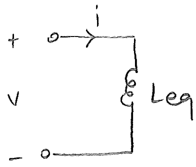
$$\Rightarrow \boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}}$$

Inductors in series connection



$i(t_0) = I_0$

$\equiv$



$i(t_0) = I_0$

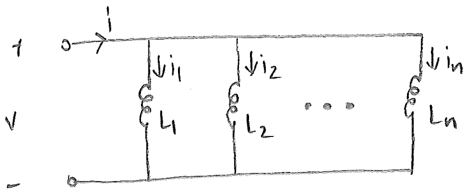
$$V = L_{eq} \frac{di(t)}{dt}$$

?

$$V = V_1 + V_2 + \cdots + V_n$$

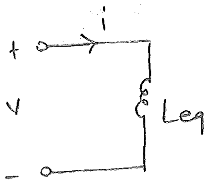
$$= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \cdots + L_n \frac{di}{dt} = \underbrace{[L_1 + L_2 + \cdots + L_n]}_{L_{eq}} \frac{di}{dt} \Rightarrow \boxed{L_{eq} = L_1 + L_2 + \cdots + L_n}$$

Inductors in parallel connection



$i_1(t_0) = I_{10}, \ldots, i_n(t_0) = I_{n0}$

$\equiv$



$i(t_0) = I_{10} + I_{20} + \cdots I_{n0}$

$$i(t) = i(t_0) + \frac{1}{L_{eq}} \int_{t_0}^{t} v(\tau)\,d\tau$$

?

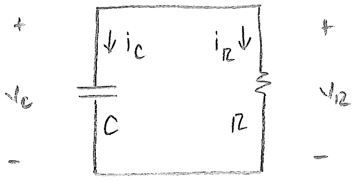
$$i(t) = i_1(t) + i_2(t) + \cdots + i_n(t)$$

$$= \left\{ I_{10} + \frac{1}{L_1} \int_{t_0}^{t} v(\tau)\,d\tau \right\} + \cdots + \left\{ I_{n0} + \frac{1}{L_n} \int_{t_0}^{t} v(\tau)\,d\tau \right\}$$

$$= \{ I_{10} + \cdots + I_{n0} \} + \underbrace{\left[\frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n} \right]}_{\frac{1}{L_{eq}}} \int_{t_0}^{t} v(\tau)\,d\tau$$

$$\Rightarrow \boxed{\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_n}}$$

<u>Zero input response</u> Response of a circuit to initial conditions only. (No input.)



$v_C(0) = V_0$

$v_C(t) = ?$ for $t \geq 0$ (Notation: $Df := \frac{d}{dt} f(t)$)

$i_C = C Dv_C$

$$\Rightarrow Dv_C = \frac{1}{C} i_C \overset{KCL}{=} -\frac{1}{C} i_R \overset{\text{terminal eqn.}}{=} -\frac{1}{C}\frac{v_R}{R} \overset{KVL}{=} -\frac{1}{RC} v_C$$

$$\Rightarrow Dv_C + \frac{1}{RC} v_C = 0 \qquad (1)$$

Eqn. (1) is a first-order homogeneous (i.e. the right-hand side is zero) differential equation. (The right-hand side is zero because we have no input.) Solution to (1) is of the form $v_C(t) = Ke^{st}$ where K and s are to be determined.

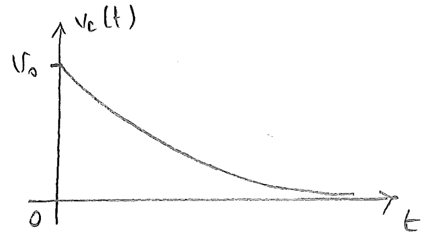
$$\frac{d}{dt}\{Ke^{st}\} + \frac{1}{RC}\{Ke^{st}\} = 0 \Rightarrow sKe^{st} + \frac{1}{RC}Ke^{st} = 0$$

$$\Rightarrow \left[s + \frac{1}{RC}\right] Ke^{st} = 0 \Rightarrow s = -\frac{1}{RC} \text{ : "natural frequency"}$$

Then $v_C(t) = Ke^{-t/RC}$ for $t \geq 0$. How about K?

Initial cond. constraint : $V_0 = v_C(0) = Ke^{-t/RC}\Big|_{t=0} = K \Rightarrow K = V_0$

Hence $\boxed{v_C(t) = V_0 e^{-t/RC}}$



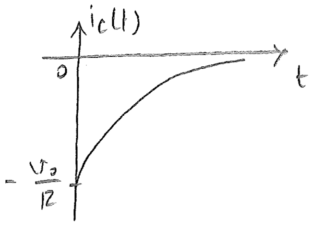
<u>Remark</u> $\tau = RC$ is sometimes called the "time constant." Note that the smaller the time constant the faster the response.

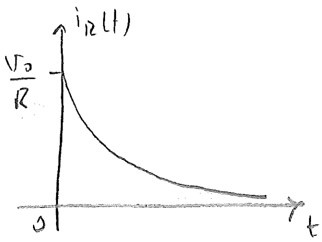
How about $i_C(t)$, $i_R(t)$?

$$i_C(t) = C\frac{d}{dt}v_C(t) = C\frac{d}{dt}\left\{V_0 e^{-t/RC}\right\} = -\frac{V_0}{R}e^{-t/RC}$$

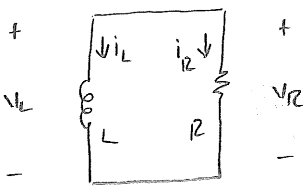
$$i_R(t) = \frac{v_R(t)}{R} = \frac{v_C(t)}{R} = \frac{V_0}{R}e^{-t/RC}$$

Note that $i_C(t) + i_R(t) = 0$ (KCL) as expected.





Example Find $i_L(t)$ for $t \geq 0$.



$i_L(0) = I_0$

$$Di_L = \frac{1}{L}v_L = \frac{1}{L}v_R = \frac{1}{L}Ri_R = -\frac{R}{L}i_L$$

$$\Rightarrow Di_L + \frac{R}{L}i_L = 0$$

Natural freq. $s = -\frac{R}{L}$ $\Rightarrow$ $i_L(t) = Ke^{-\frac{R}{L}t}$, $K = ?$

Init. cond. constraint : $i_L(0) = I_0$ $\Rightarrow$ $K = I_0$

$$\Rightarrow \boxed{i_L(t) = I_0 e^{-\frac{R}{L}t}}$$

(Sometimes we write $i_L(t) = I_0 e^{-t/\tau}$ where $\tau = \frac{L}{R}$ is the time constant.)

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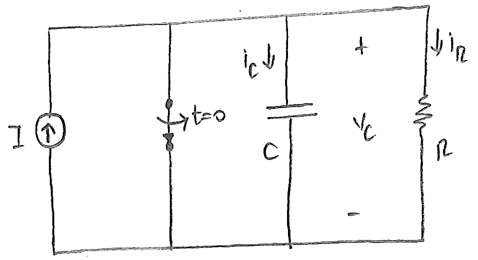
Zero-state response Response of a circuit with zero init. cond. to an input.

Example [constant input]



Note that $v_C(0) = 0$

"zero initial condition"

For $t > 0$ we can write

$$Dv_C = \frac{1}{C}i_C = \frac{1}{C}(I - i_R) = \frac{1}{C}\left(I - \frac{v_C}{R}\right)$$

$$Dv_C + \frac{1}{RC}v_C = \underbrace{\frac{I}{C}}_{\text{input term}} \quad (*)$$

Solution to (*) is of the form : $v_C(t) = \underbrace{v_h(t)}_{\text{homogeneous solution}} + \underbrace{v_p(t)}_{\text{particular solution}}$

To find $v_h(t)$ ignore the right-hand side (i.e. consider the homogeneous diff. eqn.)

$$Dv_c + \frac{1}{RC} v_c = 0 \Rightarrow \underbrace{v_h(t) = Ke^{-t/RC}}_{\text{hmg. solution}}$$

hmg. solution (K to be determined later)

Natural freq. $= -\frac{1}{RC}$ (time constant $= RC$)

to find $v_p(t)$ v_p has the same "form" as input. [ex input constant $\Rightarrow$ v_p constant]

Hence let $v_p(t) = A$ (constant)

Substitute $v_p(t)$ in the diff. eqn.

$$\left.\frac{d}{dt} v_p(t) + \frac{1}{RC} v_p(t) = \frac{I}{C}\right|_{v_p(t)=A} \Rightarrow \frac{A}{RC} = \frac{I}{C} \Rightarrow A = IR$$

$\Rightarrow v_p(t) = IR$

Now, the overall solution : $v_c(t) = v_h(t) + v_p(t) = Ke^{-t/RC} + IR$ (K yet unknown)

To find K, use the init. cond. constraint :

$$\left. Ke^{-t/RC} + IR \right|_{t=0} = v_c(0) = 0 \Rightarrow K = -IR$$

Finally, $\boxed{v_c(t) = IR\left(1 - e^{-t/RC}\right)}$