Ch. II   Linear Time-Invariant Resistive Circuits

Q: What is an LTI resistive circuit?

Def A nonlinear circuit contains a nonlinear R or L or C or dependent source (DS)

A circuit is linear if it is NOT nonlinear.

Def A time-varying circuit contains a TV R or L or C or DS.

A circuit is time-invariant if it is NOT time-varying.

Def A dynamic circuit contains an L or C.

A circuit is resistive if it is NOT dynamic.

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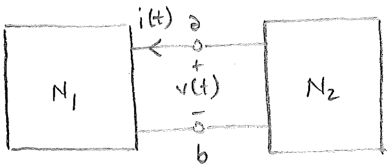
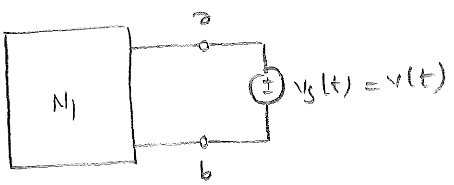
Substitution Thm. Consider the two circuits N_1, N_2 below:

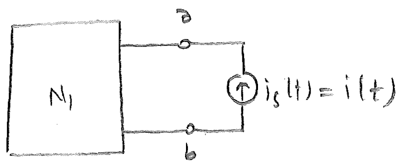


(I)

We can replace N_2 by either an independent voltage source with $v_s(t) = v(t)$ or by an independent current source with $i_s(t) = i(t)$ without affecting any branch voltage / current inside N_1.



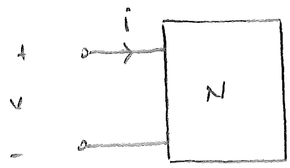
(II)



(III)

In other words, an observer inside N_1 cannot tell the difference between the configurations I, II, III.

Definition A pair of associated terminals is called a port. One-port is a circuit or circuit element that has one pair of terminals accessible from outside:

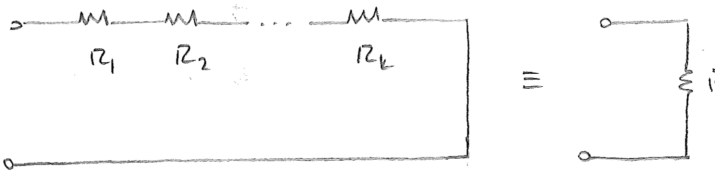


one-port circuit

Two one-ports are said to be equivalent if they have identical i-v characteristics at the port terminals. Let N_0, N_1, N_2 be one-ports and N_1 be equivalent to N_2 ($N_1 \equiv N_2$). Then for both of the following configurations the set of branch currents and voltages within N_0 will be the same.

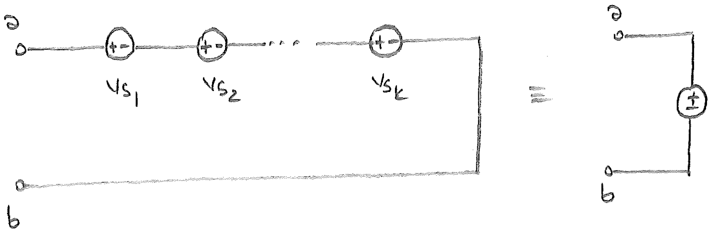
($N_1 \equiv N_2$)

Some equivalent one-ports

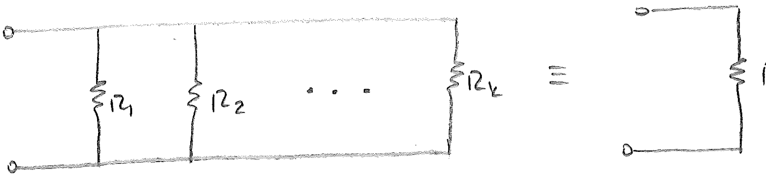


$R_{eq} = R_1 + R_2 + \cdots + R_k$

(series connection)

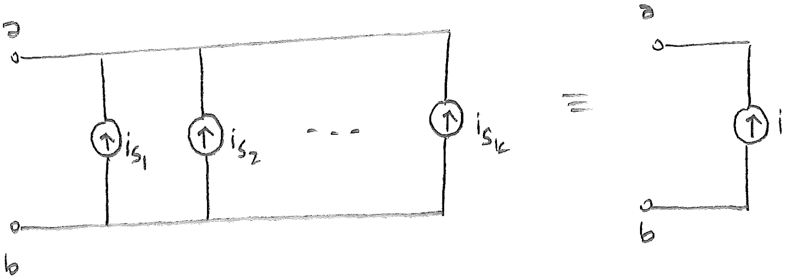


$v_{seq} = v_{s_1} + v_{s_2} + \cdots + v_{s_k}$

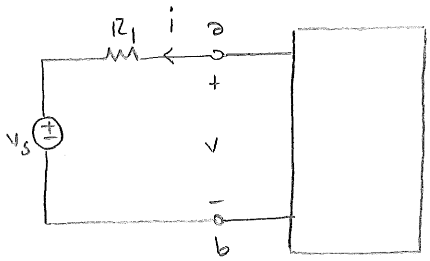


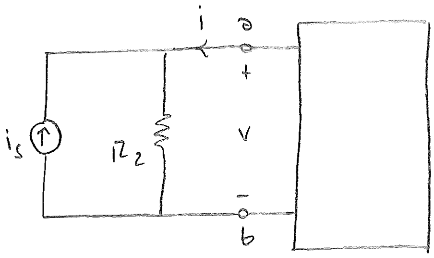
$R_{eq} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_k} \right)^{-1}$

(parallel connection)



$i_{seq} = i_{s_1} + i_{s_2} + \cdots + i_{s_k}$





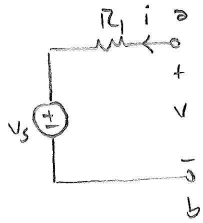
$v = R_1 i + v_s \quad (1)$

$v = R_2 (i + i_s)$

$\Rightarrow \quad v = R_2 i + R_2 i_s \quad (2)$

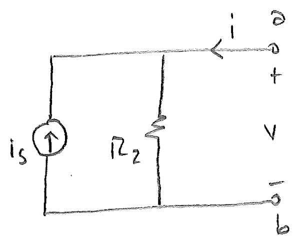
Hence

→ The one-port



has the i-v char. given in Eq. (1)

→ The one-port

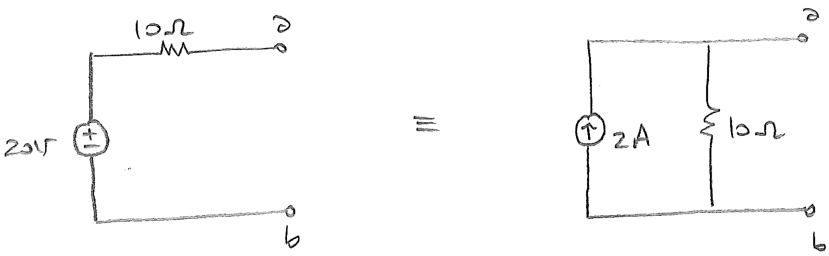


has the i-v char. given in Eq. (2)

Now, take $R_1 = R_2$ and $v_s = R_1 i_s$. Then (1) and (2) become identical.

Therefore, one-ports become equivalent.

Ex



Solving LTI resistive circuits

There are two methods:

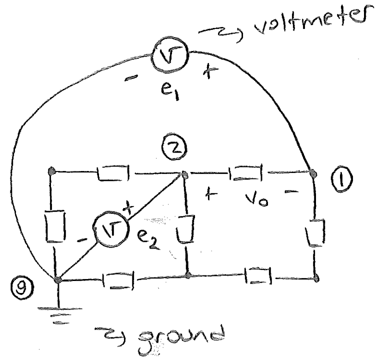
1) Node voltage analysis

2) Mesh current analysis

Definition [Suppose a reference node ⓖ (called "ground") is specified.] The **node voltage** e_k (of the node ⓚ) satisfies $e_k = v_{kg}$.

Remark Let there exist a two-terminal element between nodes ⓐ and ⓑ. The branch voltage v_{ab} associated to this element satisfies $v_{ab} = e_a - e_b$.

Ex



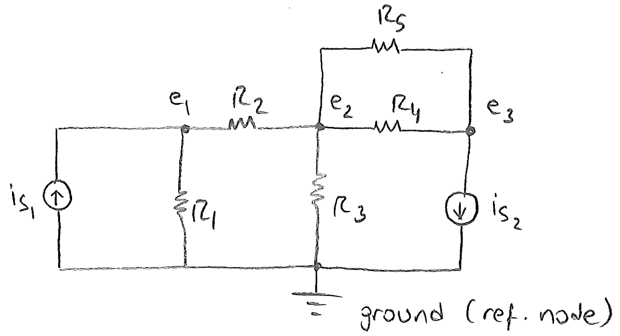
$$V_0 = e_2 - e_1$$

$$\left[V_0 = V_{21} = V_{2g} + V_{g1} = V_{2g} - V_{1g} = e_2 - e_1\right]$$

Node Voltage Analysis

The idea is to solve for the node voltages by writing KCL at each node (except the reference node). Note that once the node voltages are known, the branch voltages can be computed easily. And once the branch voltages are known, the branch currents can be computed via terminal equations.

Example



procedure:

→ Choose reference node

→ label node voltages

→ write KCL

KCL at node ①: $-i_{s_1} + i_{R_1} + i_{R_2} = 0$ $\Rightarrow -i_{s_1} + \dfrac{e_1}{R_1} + \dfrac{e_1 - e_2}{R_2} = 0 \quad (1)$

$$\frac{e_1}{R_1} \leftarrow \frac{V_{R_1}}{R_1} \qquad \frac{V_{R_2}}{R_2} \rightarrow \frac{e_1 - e_2}{R_2}$$

KCL at node ②: $\dfrac{e_2 - e_1}{R_2} + \dfrac{e_2}{R_3} + \dfrac{e_2 - e_3}{R_4} + \dfrac{e_2 - e_3}{R_5} = 0 \quad (2)$

KCL at node ③: $\dfrac{e_3 - e_2}{R_4} + \dfrac{e_3 - e_2}{R_5} + i_{s_2} = 0 \quad (3)$

In terms of conductances $G_k = 1/R_k$:

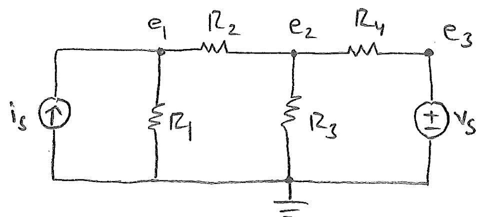
$$\left.\begin{aligned}
(1) &\Rightarrow (G_1+G_2)e_1 - G_2 e_2 = i_{s_1} \\
(2) &\Rightarrow -G_2 e_1 + (G_2+G_3+G_4+G_5)e_2 - (G_4+G_5)e_3 = 0 \\
(3) &\Rightarrow -(G_4+G_5)e_2 + (G_4+G_5)e_3 = -i_{s_2}
\end{aligned}\right\} (*)$$

$$(*) \Rightarrow \underbrace{\begin{bmatrix} G_1+G_2 & -G_2 & 0 \\ -G_2 & G_2+G_3+G_4+G_5 & -(G_4+G_5) \\ 0 & -(G_4+G_5) & G_4+G_5 \end{bmatrix}}_{Y_n \,=\, \text{node admittance matrix}} \underbrace{\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}}_{\text{node volt. vector}} = \underbrace{\begin{bmatrix} i_{s_1} \\ 0 \\ -i_{s_2} \end{bmatrix}}_{\text{source current vector}}$$

⟵ node equations in matrix form

Node analysis with voltage sources

Example



formulation variables : e_1, e_2, e_3

# of eqn. needed = 3

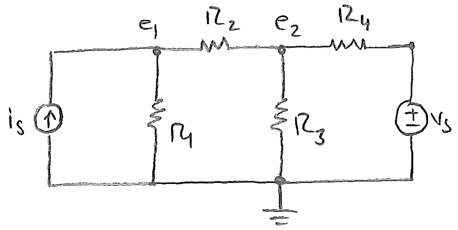
Node ① : $-i_s + \dfrac{e_1}{R_1} + \dfrac{e_1 - e_2}{R_2} = 0$

Node ② : $\dfrac{e_2 - e_1}{R_2} + \dfrac{e_2}{R_3} + \dfrac{e_2 - e_3}{R_4} = 0$

Node ③ : KCL does not give us a useful equation (in terms of e_1, e_2, e_3) here!
However, note that $e_3 = V_s$ (this is the third equation).

Remark When there is an independent voltage source V_s between the ground and some node k, we do not consider the node voltage e_k as a formulation variable (since it is NOT an unknown, $e_k = V_s$)

Example (Revisited)



formulation var. : e_1, e_2

\# of eqn. needed = 2

Node ①: $-i_s + \frac{e_1}{R_1} + \frac{e_1 - e_2}{R_2} = 0$

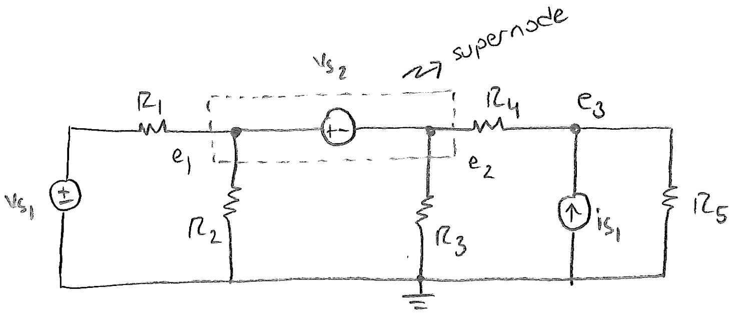
Node ②: $\frac{e_2 - e_1}{R_2} + \frac{e_2}{R_3} + \frac{e_2 - v_s}{R_4} = 0$

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_4} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} i_s \\ \frac{v_s}{R_4} \end{bmatrix}$$

Modified Node Analysis

When there is a voltage source (dependent or independent) between two (non-ground) nodes a and b, we cannot express the current through it in terms of e_a and e_b. In such cases we write KCL pretending that the two nodes (between which there is the voltage source) are a single node. Such a node is called a supernode.

Example



formulation var. e_1, e_2, e_3

Supernode Ⓢ: $\frac{e_1 - v_{s_1}}{R_1} + \frac{e_1}{R_2} + \frac{e_2}{R_3} + \frac{e_2 - e_3}{R_4} = 0$

Node ③: $\frac{e_3 - e_2}{R_4} - i_{s_1} + \frac{e_3}{R_5} = 0$

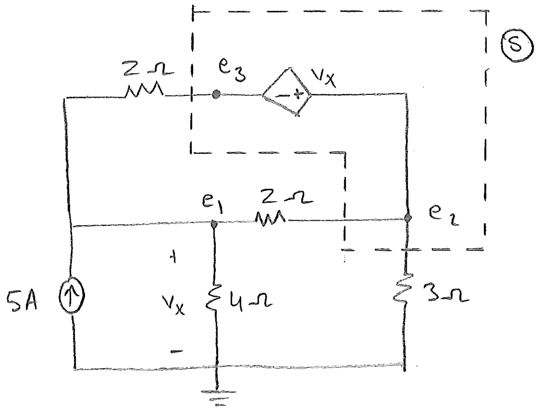
constraint : $e_1 - e_2 = v_{s_2}$

$$\underbrace{\qquad\qquad}_{\text{Node equations}}$$

$$\begin{bmatrix} G_1 + G_2 & G_3 + G_4 & -G_4 \\ 0 & -G_4 & G_4 + G_5 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} G_1 v_{s_1} \\ i_{s_1} \\ v_{s_2} \end{bmatrix}$$

$$\underbrace{\qquad\qquad}_{\text{Node eqn.'s in matrix form}}$$

Example



Find the power absorbed by the 3Ω resistor.

KCL at node ①: $-5 + \frac{e_1 - e_3}{2} + \frac{e_1}{4} + \frac{e_1 - e_2}{2} = 0$ (1)

KCL at supernode (S): $\frac{e_3 - e_1}{2} + \frac{e_2 - e_1}{2} + \frac{e_2}{3} = 0$ (2)

Constraint eqn.: $\left.\begin{array}{l} e_2 - e_3 = v_x \\ v_x = e_1 \end{array}\right\}$ $e_2 - e_3 = e_1$ (3)

Three eqn.'s three unknowns

$(1) \Rightarrow \left\{\frac{1}{2} + \frac{1}{4} + \frac{1}{2}\right\} e_1 - \frac{1}{2} e_2 - \frac{1}{2} e_3 = 5 \overset{(3)}{\Rightarrow} \frac{5}{4}(e_2 - e_3) - \frac{1}{2} e_2 - \frac{1}{2} e_3 = 5$

$\Rightarrow \frac{3}{4} e_2 - \frac{7}{4} e_3 = 5$ (4)

$(2) \Rightarrow -\left(\frac{1}{2} + \frac{1}{2}\right) e_1 + \left(\frac{1}{2} + \frac{1}{3}\right) e_2 + \frac{1}{2} e_3 = 0 \overset{(3)}{\Rightarrow} -(e_2 - e_3) + \frac{5}{6} e_2 + \frac{1}{2} e_3 = 0$

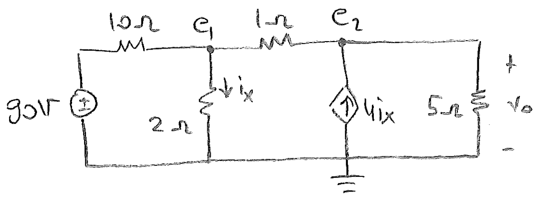
$\Rightarrow -\frac{1}{6} e_2 + \frac{3}{2} e_3 = 0 \Rightarrow e_3 = \frac{1}{9} e_2$ (5)

$(4)\&(5) \Rightarrow \frac{3}{4} e_2 - \frac{7}{4} \cdot \frac{1}{9} e_2 = 5 \Rightarrow \frac{20}{36} e_2 = 5 \Rightarrow e_2 = 9V$ $(e_3 = 1V,\ e_1 = 8V)$

$\Rightarrow P_{3\Omega} = \frac{e_2^2}{3} = \frac{9^2}{3} = \boxed{27W}$

Exercise Redo the example by choosing node ③ as ground.

Example [A.&S.]



Find v_o.

Node ①: $\dfrac{e_1-90}{10}+\dfrac{e_1}{2}+\dfrac{e_1-e_2}{1}=0 \Rightarrow \dfrac{8}{5}e_1-e_2=9 \quad (1)$

Node ②: $\left.\begin{array}{l}\dfrac{e_2-e_1}{1}-4i_x+\dfrac{e_2}{5}=0\\ i_x=\dfrac{e_1}{2}\end{array}\right\}\ \dfrac{e_2-e_1}{1}-4\left(\dfrac{e_1}{2}\right)+\dfrac{e_2}{5}=0$

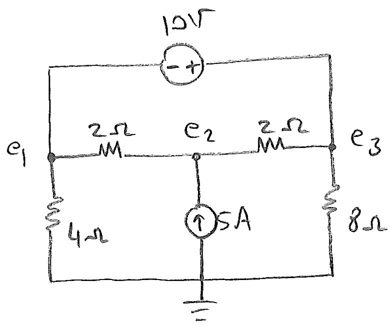
$\Rightarrow -3e_1+\dfrac{6}{5}e_2=0 \Rightarrow e_1=\dfrac{2}{5}e_2 \quad (2)$

(1)&(2) $\Rightarrow \dfrac{8}{5}\left(\dfrac{2}{5}e_2\right)-e_2=9 \Rightarrow -\dfrac{9}{25}e_2=9 \Rightarrow e_2=-25V$

$\Rightarrow \boxed{v_o=-25V}$

Example [A&S]

Write the node equations in matrix form.



Supernode ①+③: $\dfrac{e_1}{4}+\dfrac{e_1-e_2}{2}+\dfrac{e_3-e_2}{2}+\dfrac{e_3}{8}=0$

$\Rightarrow \dfrac{3}{4}e_1-e_2+\dfrac{5}{8}e_3=0 \quad (1)$

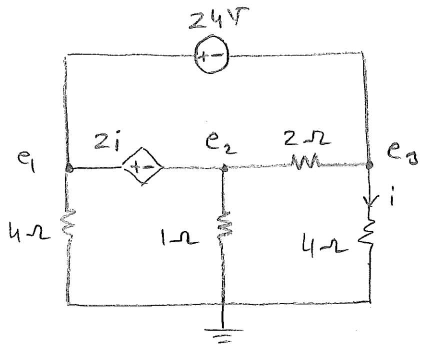
Node ②: $\dfrac{e_2-e_1}{2}-5+\dfrac{e_2-e_3}{2}=0$

$\Rightarrow -\dfrac{1}{2}e_1+e_2-\dfrac{1}{2}e_3=5 \quad (2)$

Constraint: $e_3-e_1=10 \quad (3)$

$$(1),(2),(3) \Rightarrow \begin{bmatrix} \frac{3}{4} & -1 & \frac{5}{8} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 5 \\ 10 \end{bmatrix}$$

Example



Find the power dissipated by the 1Ω resistor

Supernode ①+②+③ : $\frac{e_1}{4} + \frac{e_2}{1} + \cancel{\frac{e_2 - e_3}{2}} + \cancel{\frac{e_3 - e_2}{2}} + \frac{e_3}{4} = 0$

$\Rightarrow \frac{1}{4}e_1 + e_2 + \frac{1}{4}e_3 = 0 \quad (1)$

constraint 1: $e_1 - e_3 = 24 \Rightarrow e_1 = e_3 + 24 \quad (2)$

constraint 2: $\left.\begin{array}{l} e_1 - e_2 = 2i \\ i = \frac{e_3}{4} \end{array}\right\} e_1 - e_2 = \frac{1}{2}e_3 \Rightarrow e_1 = e_2 + \frac{1}{2}e_3 \quad (3)$

(2)&(3) $\Rightarrow e_3 + 24 = e_2 + \frac{1}{2}e_3 \Rightarrow e_2 = \frac{1}{2}e_3 + 24 \quad (4)$

(1),(2)&(4) $\Rightarrow \frac{1}{4}(e_3 + 24) + \left(\frac{1}{2}e_3 + 24\right) + \frac{1}{4}e_3 = 0 \Rightarrow e_3 = -30V \quad (5)$

(4)&(5) $\Rightarrow e_2 = \frac{1}{2}(-30) + 24 = 9V$

$\Rightarrow P_{1\Omega} = \frac{e_2^2}{1} = \boxed{81W}$