A Note on Transformers

A transformer is a device that transfers electrical energy from one circuit to another through inductively coupled conductors, i.e., the transformer's coils. A typical device is shown in Fig. 1. This note presents a simple mathematical description of this very important component in electrical networks.

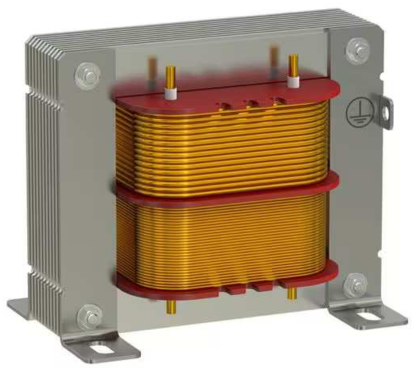

Figure 1: A transformer.

1 Single winding case

When a current i flows through a coil wound around a core, a *magnetizing flux* ϕ_{m} and a *leakage flux* ϕ_ℓ develop; see Fig. 2. The core is usually made of a ferromagnetic material, which enables generation of large magnetic fluxes by relatively low exciting currents and provides a well-defined path for the associated magnetic fields.

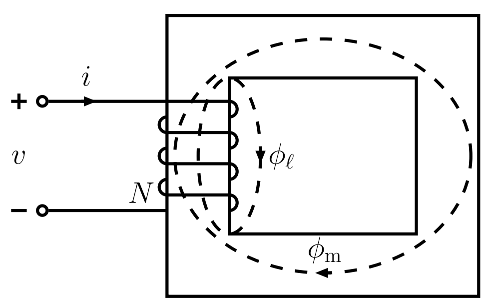

Figure 2: A core with a single winding.

Let us now derive a linear model for the magnetic circuit in Fig. 2. We begin with expressing the magnetizing and leakage fluxes as

$$\phi_{\mathrm{m}} = \wp_{\mathrm{m}} N i \tag{1a}$$

$$\phi_\ell = \wp_\ell N i \tag{1b}$$

where N is the *number of turns* in the coil, $\wp_\mathrm{m}$ the *core permeance*, $\wp_\ell$ the *leakage permeance*, and i the *current* flowing through the winding. Here, permeance can be understood as a measure of the ability of a magnetic circuit to conduct flux. It is analogous to the conductance concept in electrical circuits. As shown in Fig. 2, the magnetizing flux is confined within the core while the leakage flux completes its path outside, i.e., through the air. Since the permeability of the ferromagnetic core is much larger than that of the air, in certain applications one can safely ignore the leakage flux and consider the magnetizing flux only. Now we obtain the i-v relation in our magnetic circuit, where v is the *voltage* across the terminals of the coil.

The total flux linking each turn of the coil is

$$\phi = \phi_\mathrm{m} + \phi_\ell \tag{2}$$

which gives rise to the *flux linkage** λ with

$$\lambda = N\phi\,. \tag{3}$$

Combining (1), (2), and (3) with Faraday's law

$$v = \frac{d\lambda}{dt}$$

yields

$$v = N^2(\wp_\mathrm{m} + \wp_\ell)\frac{di}{dt} = L\frac{di}{dt}$$

where $L = N^2(\wp_\mathrm{m} + \wp_\ell)$ is the *inductance* of the coil in Fig. 2.

2 Double winding case

Now we extend our previous analysis of the single winding device to the double winding case; see Fig. 3.

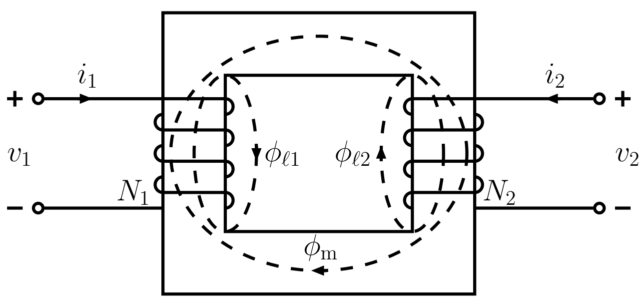

Figure 3: A core with two windings.

The extension is quite straightforward. We begin with fluxes. The winding currents i_1 and i_2 jointly contribute to the magnetic flux (ϕ_m) circulating within the core while they generate their own leakage fluxes ($\phi_{\ell 1}$ and $\phi_{\ell 2}$) individually. More formally, we have†

$$\phi_\mathrm{m} = \wp_\mathrm{m}(N_1 i_1 + N_2 i_2) \tag{4a}$$
$$\phi_{\ell 1} = \wp_\ell N_1 i_1 \tag{4b}$$
$$\phi_{\ell 2} = \wp_\ell N_2 i_2\,. \tag{4c}$$

*The careful student will readily notice that what we call *flux linkage* here is called *flux* in the courses EE201 and EE202.

†For simplicity we take $\wp_{\ell 1} = \wp_{\ell 2} = \wp_\ell$ here.

And these fluxes in turn produce the flux linkages λ_1 and λ_2 (associated to the primary and secondary windings, respectively) as follows

$$\lambda_1 = N_1(\phi_{\rm m} + \phi_{\ell 1}) \tag{5a}$$
$$\lambda_2 = N_2(\phi_{\rm m} + \phi_{\ell 2})\,. \tag{5b}$$

Finally, invoking Faraday's law $v_k = d\lambda_k/dt$ $(k = 1,\, 2)$ in the light of (4) and (5), we obtain the terminal equations of our coupled coils

$$v_1 = [N_1^2(\wp_{\rm m} + \wp_\ell)]\frac{di_1}{dt} + [N_1N_2\wp_{\rm m}]\frac{di_2}{dt} \tag{6a}$$
$$v_2 = [N_1N_2\wp_{\rm m}]\frac{di_1}{dt} + [N_2^2(\wp_{\rm m} + \wp_\ell)]\frac{di_2}{dt}\,. \tag{6b}$$

Observe that the above pair of terminal equations yield the following inductance matrix $L \in \mathbb{R}^{2\times 2}$

$$L = \begin{bmatrix} N_1^2(\wp_{\rm m} + \wp_\ell) & N_1N_2\wp_{\rm m} \\ N_1N_2\wp_{\rm m} & N_2^2(\wp_{\rm m} + \wp_\ell) \end{bmatrix}$$

which is clearly symmetric. Also,

$$\det(L) = N_1^2N_2^2\wp_\ell(2\wp_{\rm m} + \wp_\ell) \geq 0 \tag{7}$$

is never negative, verifying the fact that what we study here is a passive device.

3 Ideal transformer

As mentioned earlier, the permeability of the ferromagnetic core is much larger than that of the air. A result of this is $\wp_{\rm m} \gg \wp_\ell$. Sometimes it is excusable to take this relation to its limit, i.e.,

$$\wp_\ell \to 0 \quad \text{and} \quad \wp_{\rm m} \to \infty \tag{8}$$

which simplifies the mathematical model without significantly compromising its accuracy with respect to the actual device, provided that it is operating under certain favorable conditions. The hypothetical device thus obtained is called the *ideal transformer*. Using the results of our earlier analysis we now derive the terminal equations of the ideal transformer

$$\frac{v_1}{N_1} = \frac{v_2}{N_2} \quad \text{and} \quad N_1i_1 + N_2i_2 = 0$$

under the assumption (8). When $\wp_\ell = 0$ the determinant (7) becomes zero. This case (i.e., $\det(L) = 0$) corresponds to the so called *perfect coupling* of the coils. Under perfect coupling the inductance matrix L is rank deficient, meaning its range space is of dimension one. Note that the voltage vector $[v_1\ v_2]^T$ belongs to this range space by (6). Hence the ratio v_1/v_2 has to be constant under perfect coupling. This ratio readily follows from (6) under $\wp_\ell = 0$:

$$\frac{v_1}{v_2} = \frac{[N_1^2\wp_{\rm m}]di_1/dt + [N_1N_2\wp_{\rm m}]di_2/dt}{[N_1N_2\wp_{\rm m}]di_1/dt + [N_2^2\wp_{\rm m}]di_2/dt} = \frac{N_1\wp_{\rm m}}{N_2\wp_{\rm m}} \times \frac{N_1di_1/dt + N_2di_2/dt}{N_1di_1/dt + N_2di_2/dt} = \frac{N_1}{N_2}$$

as expected. And to get the current equation of the ideal transformer we consider (4a) under $\wp_{\rm m} \to \infty$:

$$N_1i_1 + N_2i_2 = \lim_{\wp_{\rm m}\to\infty} \frac{\phi_{\rm m}}{\wp_{\rm m}} = 0$$

which was to be shown.

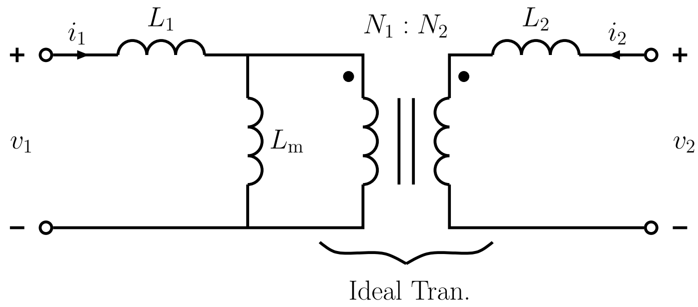

Figure 4: A two-port circuit satisfying (6).

4 Equivalent circuit model

An electrical circuit, comprising three inductors and an ideal transformer, that fits the model (6) is given in Fig. 4.

Problem. Find the inductance values L_1, L_2, L_{m} in the equivalent circuit of Fig. 4 so that (6) is satisfied.

Answer. $L_1 = N_1^2 \wp_\ell$, $L_2 = N_2^2 \wp_\ell$, $L_{\mathrm{m}} = N_1^2 \wp_{\mathrm{m}}$.

An improvement on our model is presented in Fig. 5. In this more realistic circuit the resistances R_1 and R_2 represent the copper losses in the primary and secondary windings, respectively; and the resistance R_{c} accounts for the core losses jointly caused by the hysteresis effect and the eddy currents. The hysteresis-based loss stems from the nonlinear relation between the current and the generated magnetic flux. (Recall that we have initially assumed this relation to be linear.) And eddy currents develop in the core in order to oppose the change in the flux. To compensate the demagnetizing effect of eddy currents, the exciting current should increase.

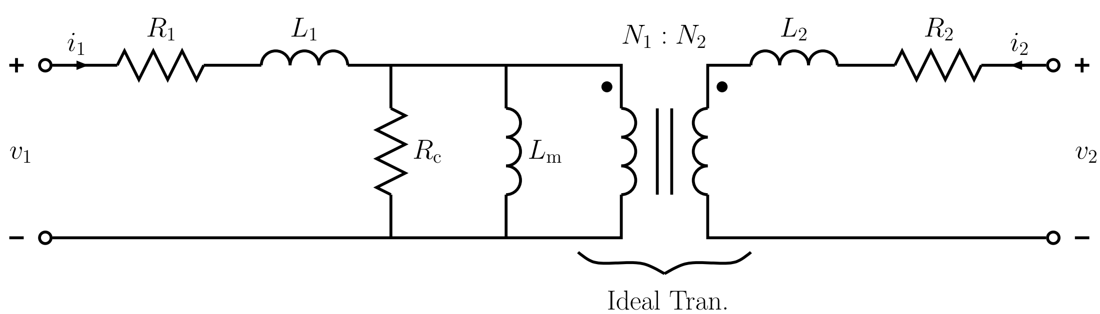

Figure 5: A more realistic model taking the copper and core losses into consideration.

References

[1] A.E. Fitzgerald, C. Kingsley, & S.D. Umans (1983). *Electric Machinery (4th ed.)* McGraw-Hill.

[2] Yıldırım Üçtuğ (2004). *EE361 Lecture Notes.* METU.