Report for the Modified Theories of Gravity

Inflationary Universe

Kerim Demirel
demirelk@metu.edu

Abstract

In this report, after summarizing the language of modern cosmology, the idea behind cosmic inflation will be introduced based on Alan Guth's seminal paper published in 1981[1]. I'll show that how the inflation, which postulates a period of accelerated expansion during the Universe's earliest stages, solves the problems of the old big bang theory. And then, I'll discuss how to model the inflation using scalar fields or special fluids, whose equation of state is different from standard matter and radiation. In addition, modified gravity theories will also be mentioned in the context of modeling the inflation.

1. The language of modern cosmology

The standard big bang theory is based on the cosmological principle, which states that the Universe looks the same to all observers. This statement put a constrain on the Universe that it must be homogeneous and isotropic, which can be described by the Friedmann-Lemaitre-Robertson–Walker (FLRW) metric of the form[2]

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{1}$$

in natural units where $c = \hbar = 1$. Here, t is the time variable, and $r - \theta - \phi$ are spatial comoving (polar) coordinates. Given an object at rest on the coordinates (r, θ, ϕ) remains the same with the expansion while the physical distances are related with comoving distances by

$$\text{physical distance} = a(t) \times \text{comoving distance.}$$

So, the function $a(t)$ is called the scale factor, and basically, it measures the spatial expansion of the Universe. On the other hand, the constant parameter k measures the spatial curvature, and with rescaling the coordinates, it takes one of the three discreet values $\{-1, 0, 1\}$ corresponding to (hyperbolic) open, flat, and (spherical) closed Universes respectively.

After substituting the metric (1) into the Einstein's equation,

$$G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2}g^\mu_\nu R = 8\pi G\, T^\mu_\nu \tag{2}$$

we can get the equations of motion. For the simplest case, if we model the energy content of the universe as a homogeneous perfect fluid, the energy-momentum tensor, T^μ_ν, takes the form $\text{diag}(-\rho, p, p, p)$ where $\rho(t)$ gives the energy density and $p(t)$ is the pressure at given time t. In that case, from $(0,0)$ and (i,i) components of the Einstein's equation (where $i = 1, 2, 3$), the EoM can be obtained as

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2(t)}, \qquad \text{(Friedmann Equation)} \tag{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \,, \qquad \text{(Acceleration Equation)} \tag{4}$$

where H is defined as $H(t) := \frac{\dot{a}}{a}$ and called "Hubble parameter". In addition, from the energy conservation, namely the vanishing of the covariant derivative of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, we have the following relation, named fluid equation,

$$\dot{\rho} + 3H(\rho + p) = 0. \qquad \text{(Fluid Equation)} \tag{5}$$

Together with (3) and (4), these three equations are fundamental equations of modern cosmology. So, our aim is then trying to find the evolution of energy density $\rho(t)$ and scale-factor $a(t)$ with time for a given equation of state (EoS) for the content of the Universe.

For example, in flat geometry where $k = 0$, for matter, radiation and cosmological constant dominated Universes separately, whose EoS are $p = 0$, $p = \rho/3$ and $p = -\rho$, respectively, we have the following solutions for $\rho(a)$ and $a(t)$ as in the table below.

Content	EoS	Energy Density	Scale Factor
Radiation:	$p = \rho/3$	$\rho \propto a^{-4}$	$a(t) \propto t^{1/2}$
Matter:	$p = 0$	$\rho \propto a^{-3}$	$a(t) \propto t^{2/3}$
Λ :	$p = -\rho$	$\rho = \text{constant}$	$a(t) \propto \mathrm{e}^{Ht}$

Moreover, it is useful to define two important parameters which are critical density ρ_c and the density parameter $\Omega(t)$. The former defines the necessary density for the Universe to be flat, and can be expressed as

$$\rho_c = \frac{3H^2}{8\pi G},$$

while the latter is defined as the ratio of the energy density to the critical density at given time t,

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_c}.$$

Then, the Friedmann equation (3) can be re-expressed in terms of the density parameter as

$$\Omega - 1 = \frac{k}{a^2 H^2} \,. \tag{6}$$

Note that for the mixed energy content, the density parameter can be separated as $\Omega = \Omega_m + \Omega_r + \Omega_\Lambda$, and in fact, today's observation suggested that the Universe is composed of approximately 70% of dark energy (Λ), 26% of matter (including dark matter), and 4% of radiation.

Thermodynamics of adiabatically expanding radiation dominated Universe

If we consider that the expansion is adiabatic, then from the first law of thermodynamics, $dU = TdS - pdV$, we have,

$$\frac{d}{dt}\left(\rho a^3\right) = -p\frac{d}{dt}\left(a^3\right) \,, \tag{7}$$

$$\frac{d}{dt}\left(s a^3\right) = 0 \,, \tag{8}$$

where s is the entropy density so that the entropy S is conserved.

From the Planck's energy distribution formula for the black body radiation, it can be shown that temperature and the expansion is related by $T \propto \frac{1}{a}$, so that the Universe cools as it expands. Hence, it is reasonable to think that the Universe may have been arbitrarily hot and dense in its earliest stages. That means that in the earliest times where the temperature is very high, the Universe is dominated by radiation which includes both photons and ultra-relativistic all particles whose equation of state can be approximated by that of an ideal quantum gas of massless particles. In that case, the thermodynamical functions can be given by [1:1][3]

$$\rho = 3p = \frac{\pi^2}{30}\mathfrak{N}(T)T^4 \;, \tag{9}$$

$$s = \frac{2\pi^2}{45}\mathfrak{N}(T)T^3 \;, \tag{10}$$

$$n = \frac{\zeta(3)}{\pi^2}\mathfrak{N}'(T)T^3 \;, \tag{11}$$

where $\zeta(3) \approx 1.202$ is the value of Riemann zeta function at 3. The functions $\mathfrak{N}(T)$ and $\mathfrak{N}'(T)$ can be expressed in terms of the number of bosonic spin degrees of freedom N_b which are effectively massless at temperature T and the corresponding number for fermions N_f by

$$\mathfrak{N}(T) = N_b(T) + \frac{7}{8}N_f(T) \;, \tag{12}$$

$$\mathfrak{N}'(T) = N_b(T) + \frac{3}{4}N_f(T) \;. \tag{13}$$

Hence, assuming that T is not near any mass thresholds, the Friedmann equation (3) can be re-expressed in terms of temperature as

$$\left(\frac{\dot{T}}{T}\right)^2 + \epsilon(T)T^2 = \frac{4\pi^3}{45}G\mathfrak{N}(T)T^4 \;, \tag{14}$$

where the function $\epsilon(T)$ is defined by

$$\epsilon(T) = \frac{k}{a^2T^2} = k\left[\frac{2\pi^2}{45}\frac{\mathfrak{N}(T)}{S}\right]^{2/3} \;. \tag{15}$$

I will use these functions $(9)-(15)$ when the problems with the hot Big Bang theory are introduced.

2. Problems with the Big Bang

Like the cosmic microwave background radiation (CMB), detected by Arno Penzias and Robert Wilson in 1965, and the cosmic abundance of the light nuclear isotopes such as hydrogen, deuterium, helium-3 and helium-4, many other independent evidence support the big bang theory. However, despite the success of the old theory, it has some discrepancies deeply related to the initial conditions.

As examples, observations suggests that the Universe is very homogeneous and flat today. In fact, the level of anisotropies in CMB map is at $T/T_0 \sim 10^{-5}$ and today's value of the density parameter is about $\Omega_0 = 1 \pm 0.02$. I will show how these observations causes problems.

2.1. The flatness problem

From the Friedmann equation (6) in terms of the density parameter, we can simply write,

$$|\Omega - 1| = \frac{|k|}{a^2 H^2}\,. \tag{16}$$

Notice that the LHS of this equation measures how much the density parameter Ω close to 1, and $\Omega = 1$ defines the flat geometry. The RHS, however, is always an increasing function of time both for matter and radiation.

$$\text{Matter domination:} \quad |\Omega - 1| \propto t^{2/3}$$

$$\text{Radiation domination:} \quad |\Omega - 1| \propto t$$

So, $\Omega = 1$ is an unstable critical point, which means any deviation from 1 will increase with time.

Today $\Omega_0 = 1 \pm 0.02$ is certainly within an order of magnitude of one, and the age of the Universe is about order of 10^{17} sec, which suggests that in early times $|\Omega - 1|$ must be extremely small. For simplicity, just assuming radiation domination lasts up to today, gives

$$\begin{aligned} \text{nucleosynthesis } (t \sim 1\text{sec}): &\quad |\Omega - 1| < \mathcal{O}(10^{-16}) \\ \text{electro-weak scale } (t \sim 10^{-11}\text{sec}): &\quad |\Omega - 1| < \mathcal{O}(10^{-27}) \end{aligned}$$

This result tells us that the Universe should have been way different if it did not start with an almost exact flat geometry.

Let us examine the problem using thermodynamical functions as in A. Guth's original paper[1:2]. Since we consider the expansion is adiabatic, the entropy S is conserved quantity, so its initial value can be determined or at least bounded by current observations. Assume roughly today's value of density is $\rho_0 < 10\rho_c$; then, from Eq. (16), we have

$$\left|\frac{k}{a^2}\right| < 9H^2,$$

for today. In general case ($k = \pm 1$), this gives $a > \frac{1}{3}H^{-1} \sim 3 \times 10^9$ years. By taking the present photon temperature as $T_\gamma = 2.7\,K$ from the CMB spectrum, and using Eq. (10), it can be shown that the photon contribution to entropy is bounded by

$$S\gamma > 3 \times 10^{85}.$$

Also, by taking the neutrinos and other effectively massless particles into account, the boundary to the total entropy can be given as,

$$S > 10^{86},$$

so, using this and from Eq. (15), the value of the function ϵ is now

$$|\epsilon| < 10^{-58}\mathfrak{N}^{2/3}.$$

Therefore, using the density function (9), we can express the closeness to the flatness as,

$$\left|\frac{\rho - \rho_{\text{cr}}}{\rho}\right| = \frac{45}{4\pi^3}\frac{M_P^2}{\mathfrak{N}T^2}|\epsilon| < 3 \times 10^{-59}\mathfrak{N}^{-1/3}(M_P/T)^2, \tag{17}$$

where M_P is the Planck's mass. For the very early times where the temperature is at $T = 10^{17}\,GeV$, we have in the regime where the effective theory is grand unified theories (GUTs), and $\mathfrak{N} \sim 10^2$ is typical of grand unified models. Therefore, we see that the Universe must be extremely close to the flatness at those times, namely

$$\left|\frac{\rho-\rho_{\rm cr}}{\rho}\right| < 10^{-55}.$$

This is the flatness problem.

2.2. The horizon problem

According to CMB observations, light coming from different regions of the sky seems to be at the same average temperature (about $\sim 2.7\,K$), meaning that the Universe is very homogeneous. However, when you consider that the light have a limited distance to travel since its emission due to the finite speed of light, it came out that different regions of the sky would have never be in contact with each other in the past according to the evolution scenario in the hot big bang theory. This is a contradiction.

This problem can be shown quantitatively. A light pulse beginning at $t = 0$ will have traveled a maximum physical distance

$$l_{\rm max}(t) = a(t)\int_0^{r_{\rm max}(t)} \frac{dr}{\sqrt{1-kr^2}} = a(t)\int_0^t \frac{dt'}{a(t')}\,, \tag{18}$$

until the time t, which is called the horizon. We will compare this horizon distance with the radius $L(t)$ of the region at time t which will evolve into our observed region of the Universe. Now, due to the fact that $\epsilon(T)$ is extremely small, if we ignore the ϵT^2 term from Friedmann equation (14), the differential equation can be solved for temperature T,

$$T^2 = \frac{M_P}{2\gamma t}$$

where $\gamma^2 = (4\pi^3/45)\mathfrak{N}$. For the minimal SU_5 grand unified model, $N_b = 82$, $N_f = 90$, so that $\gamma = 21.05$.

From the conservation of entropy, notice that $aT = \textbf{const}$, so that $a \propto t^{1/2}$. Therefore, Eq. (18) becomes,

$$l(t) = a(t)\int_0^t \frac{dt'}{a(t')} = 2t\,, \tag{19}$$

which gives the physical horizon distance.

In order the Universe to have a thermal equilibrium, this particle horizon should be in the same order with the radius of the observable Universe, $L(t)$; however, from the conservation of entropy again, we can write,

$$L(t) = \left(\frac{s_p}{s(t)}\right)^{1/3} L_p\,, \tag{20}$$

where s_p is the present entropy density and the current radius of observable Universe can be taken as $L_p \sim 10^{10}$ **years**. Hence, if we calculate the ratio of volumes,

$$\frac{l^3}{L^3} = \frac{11}{43}\left(\frac{45}{4\pi^3}\right)^{3/2}\mathfrak{N}^{-1/2}\left(\frac{M_P}{L_p T_\gamma T}\right)^3 = 4\times 10^{-89}\mathfrak{N}^{-1/2}(M_P/T)^3\,. \tag{21}$$

Therefore, by taking $\mathfrak{N} \sim 10^2$ and $T = 10^{17}\ GeV$, we found

$$l_0^3/L_0^3 = 10^{-83} ,$$

which means that there should be 10^{83} regions that are causally disconnected from each other in the observable Universe. This is called the horizon problem.

3. Idea of inflation

In order to avoid these problems, Alan Guth, in 1981, proposed an idea which is called the inflation. According to this, instead of an adiabatic expansion, suppose that the Universe has a short entropy generating period at initial where the entropy is greatly increased as

$$S_p = Z^3 S_i , \tag{22}$$

where S_p and S_i denote the present and initial values of entropy ($S = a^3 s$), and Z is some large factor. This simple assumption solves the flatness and horizon problems.

Let us re-examine these problems with the assumption (22). The RHS of Eq. (17) has a Z^2 term now due to the function $\epsilon(T)$ in (15). Then, the initial value of $|\rho - \rho_{\rm cr}|/\rho$ with $T_i = 10^{17}\ GeV$ could be of order unity,

$$\left| \frac{\rho - \rho_{\rm cr}}{\rho} \right| = 3 \times 10^{-59} \mathfrak{N}^{-1/3} Z^2 \left(\frac{M_P}{T} \right)^2 \approx 1$$

if

$$Z > 3 \times 10^{27} . \tag{23}$$

Then, the flatness problem is solved.

As a solution to the horizon problem, due to assumption (22), the RHS of (20) is multiplied by factor Z^{-1}, so that if Z is sufficiently large, then the initial region which evolved into our observable Universe would have been smaller than the horizon distance at that time. Hence, the ratio of the volumes (21) is multiplied by factor Z^3,

$$\frac{l^3}{L^3} = 4 \times 10^{-89} \mathfrak{N}^{-1/2} (M_P/T)^3 Z^3 .$$

Therefore, $l^3/L^3 \approx 1$, if

$$Z > 5 \times 10^{27} . \tag{24}$$

In that case, the horizon problem is disappears. Notice that Eqns. (23) and (24) are approximately equal, since they both correspond roughly to S_i of order unity.

In the next section, I will try to describe the mechanism behind this entropy generation.

4. Modelling the inflation

In this section, I'll show how we can model the inflation, namely the exponential-like expansion at the very early stage. Starting with the Guth's model, we'll see different formalisms currently studied in the literature.

4.1. Old Inflation

In the Guth's seminal paper[1:3], he introduces a scenario which is capable of such a large entropy production. In short, he shows that if the equation of state for matter exhibits a first-order phase transition at some critical temperature T_c, then during this phase transition we have a desired expansion to solve the big bang problems. He named named this scenario as the inflationary universe and describes it as the following.

Suppose the matter exhibits a first-order phase transition at a critical temperature, T_c. As the universe cools to T_c, bubbles of the low-temperature phase nucleate and grow. If the nucleation rate for this phase transition is rather low, the Universe will continue to cool as it expands.Therefore, it supercools to some temperature, T_s which is many orders of magnitude below T_c. When the phase transition finally takes place at temperature T_s, the latent heat $Q \approx k_B T_c$ is released. Due to this heat, the universe is then reheated to some temperature T_r which is comparable to T_c as order. Assuming that the number $\mathfrak{N}$ of degrees of freedom for the two phases are comparable, the entropy density is then increased by a factor of roughly $(T_r/T_s)^3$,

$$s_f \propto \left(\frac{T_r}{T_s}\right)^3 s_i \ .$$

Hence, the large factor Z can be read as,

$$Z \approx \frac{T_r}{T_s} \ ,$$

which suggests that if the universe supercools by 28 or more orders of magnitude below the critical temperature, the horizon and flatness problems disappear.

Let us investigate the properties of this supercooling process a little bit further. As the temperature cools down to zero, $T \to 0$, the system is cooling not toward the true vacuum, but rather toward some metastable false vacuum with an energy density ρ_0; which is necessarily higher than that of the true vacuum. In that case, the equation of density (9) is now modified as,

$$\rho = \frac{\pi^2}{30}\mathfrak{N}(T)T^4 + \rho_0 \ , \tag{25}$$

which corresponds to a small modification of the Friedmann equation (14) as well.

$$\left(\frac{\dot{T}}{T}\right)^2 = \frac{4\pi^3}{45}G\mathfrak{N}(T)T^4 - \epsilon(T)T^2 + \frac{8\pi G}{3}\rho_0 \tag{26}$$

This equation has two solutions depending on the parameter ϵ.

If $\epsilon > \epsilon_0$, where

$$\epsilon_0 \equiv \frac{8\pi^2 G\sqrt{30}}{45}\sqrt{\mathfrak{N}\rho_\circ} \ ,$$

then the expansion of the Universe has stopped at some temperature T_{min} which is of $\mathcal{O}(T_c)$, and then the Universe contracts again, which is the undesired scenario.

For $\epsilon < \epsilon_0$ case, only $\epsilon < 0$ is physically plausible which correspond to the open universe. Once the temperature is low enough, the dominant term in RHS of Eq. (26) is ρ_0; therefore, we have

$$T(t) \propto e^{-\chi t} \tag{27}$$

where

$$\chi^2 = \frac{8\pi G}{3}\rho_0 \,. \tag{28}$$

Hence, from $aT = \mathbf{const}$, we have,

$$a(t) \propto e^{\chi t} \,, \tag{29}$$

meaning that the Universe is expanding exponentially, in a false vacuum state with energy density ρ_0. The Hubble parameter is given by $H \equiv \dot{a}/a = \chi$, or more precisely, H approaches χ monotonically from above.

In literature, this scenario is described by scalar fields. Scalar fields could get caught in a local minimum of the potential, which in Guth's work corresponded to a state with an unbroken grand unified symmetry. The inflation occurs due to a delayed first-order phase transition, in which a scalar field was initially trapped in a local minimum of some potential, and then leaked through the potential barrier and rolled toward a true minimum of the potential via tunnelling.

However, shortly after its publication, it is realized that this scenario has some problems.[4] For example, the transition from “false vacuum” to the lower energy “true vacuum” could not have occurred everywhere simultaneously, but here and there in small bubbles of true vacuum, which rapidly expanded into the background of false vacuum, in which the scalar field would have been still trapped in its local minimum. Therefore, the latent heat released in the phase transition would have wound up in the bubble walls, leaving the interiors of the bubbles essentially empty, so that the only places where there would be energy that could grow into the present contents of the universe would be highly inhomogeneous and anisotropic.

Because of these problems, this idea is called "old inflation", today.

4.2. Using Scalar Fields

The solution to the problems of old inflation came with the “new inflation” theory in 1981-1982 by Linde[5] and Coleman & Weinberg[6]. In the new theory, inflation starts in an unstable state at the top of the effective potential, and then the field ϕ, called inflaton, slowly rolls down to the minimum potential. The density perturbations are produced during slow-roll and inversely proportional to $\dot{\phi}$ which later become the seeds of galaxy formations. The consequences of the new inflationary theories turned out to depend on the "slow-roll" of the scalar field, rather than the process of bubble formation itself.

Now, let us examine the simple scalar field models to show how inflation occurs with this formalism. Suppose that we have a scalar field ϕ with a general action given by

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] \tag{30}$$

where g is the determinant of the metric, $g \equiv \det(g_{\mu\nu})$, the metric being flat for simplicity as $g_{\mu\nu} = \mathrm{diag}\big(1, -a^2(t), -a^2(t), -a^2(t)\big)$.

Then, the solution to the Euler-Lagrange equations, the equation of motion can be derived as

$$g^{\mu\nu}\partial_\mu\partial_\nu\phi + 3H\dot{\phi} + V'(\phi) = 0 \,. \tag{31}$$

We will be particularly interested in the homogeneous mode of the field ϕ, for which the gradient $\nabla\phi = 0$. In this case, the solution is reduced to

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0\,. \tag{32}$$

The stress-energy tensor for a scalar field is given by

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}\mathcal{L}_\phi \tag{33}$$

where $\mathcal{L}_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$. Then, from $T^\mu_\nu = g^{\mu\rho}T_{\rho\nu} = g^{\mu\rho}\partial_\rho\phi\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}_\phi$, we have

$$\rho = +T^0_0 = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{34}$$

$$p = -T^i_i = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{35}$$

We see that the de Sitter limit, $p \simeq -\rho$ occurs if the potential energy dominates the kinetic energy, namely $\dot{\phi}^2 \ll V(\phi)$. This limit is called slow-roll under which the universe expands quasi-exponentially. Therefore, We can rewrite the Friedmann and acceleration equations (3) and (4) as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{1}{3m^2_{\mathrm{Pl}}}\left[\frac{1}{2}\dot{\phi}^2 + V\right], \tag{36}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \equiv H^2(1-\epsilon)\,, \tag{37}$$

where the parameter ϵ measures the accelerated expansion as $\ddot{a} > 0 \Longleftrightarrow \epsilon < 1$, and the Planck mass is defined as $m_{\mathrm{Pl}} = 1/\sqrt{8\pi G}$. From these two equations it can be derived as

$$\epsilon = \frac{3}{2}\left(1 + \frac{p}{\rho}\right). \tag{38}$$

The de Sitter limit where $p = -\rho$ is equivalent to $\epsilon = 0$, so we have

$$H^2 \simeq \frac{1}{3m^2_{\mathrm{Pl}}}V(\phi)\,. \tag{39}$$

We will make an additional approximation that the friction term in the equation of motion $\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$ is dominating, i.e. $3H\dot{\phi} \gg \ddot{\phi}$, so that the equation of motion for the scalar field is approximately

$$3H\dot{\phi} + V'(\phi) \simeq 0\,. \tag{40}$$

This approximation together with Eq. (38) is called slow-roll approximation. Further, the condition $3H\dot{\phi} \gg \ddot{\phi}$ can be expressed in terms of another parameter η defined as

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}\,, \tag{41}$$

and together with

$$\epsilon = \frac{1}{2m^2_{\mathrm{Pl}}}\left(\frac{\dot{\phi}}{H}\right)^2, \tag{42}$$

these are called slow-roll parameters. The slow-roll approximation holds when the slow-roll parameters are very small: $\epsilon, |\eta| \ll 1$. By using equations (39) and (40), we can express these parameters in terms of the potential as

$$\epsilon = \frac{m_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2 , \tag{43}$$

$$\eta = m_{\rm Pl}^2\left[\frac{V''}{V} - \frac{1}{2}\left(\frac{V'}{V}\right)^2\right] . \tag{44}$$

Moreover, we can define the number of e-folds N with the sign convention

$$a(t) \propto e^{\int H dt} \equiv e^{-N} , \tag{45}$$

$$dN = -H dt , \tag{46}$$

so that N increases as we go back in time. So, we can re-express the number of e-fold N in terms of the potential as

$$N = -\int H dt \simeq \frac{1}{m_{\rm Pl}^2}\int_{\phi_f}^{\phi}\frac{V(\phi)}{V'(\phi)}d\phi , \tag{47}$$

where ϕ_f denotes the field at the end of inflation.

Depending on the form of the potential, there are many models proposed. As examples, $V(\phi)$ can be chosen as $\lambda\phi^4$, $\lambda(\phi^2 - M^2)^2$, $m^2\phi^2$, etc. For given potential, the calculation of slow-roll parameters are trivial.

Lastly, I would like to note that there are two observable parameters depending on the slow-roll parameters, which are called the spectral index n_s and the tensor-to-scalar ratio r. These can be given as,

$$n_s = 1 - 6\epsilon + 2\eta , \tag{48}$$

$$r = 16\epsilon . \tag{49}$$

According to the recent observations, $n_s = 0.9649 \pm 0.0042$ at 68% CL [PLANCK 2018][7]:, and $r < 0.044$ [combined analysis of PLANCK(2018) and BICEP2/Keck(2015)][8]. Therefore, model's free parameters (such as coupling constant λ) can be fitted to some numerical values using these observable parameters.

In conclusion, we can picture the scalar field driven inflation as the following. At early times, the energy density of the universe is dominated by ϕ which is slowly evolving on a nearly constant potential that approximates a cosmological constant. Inflation ends as the potential steepens and the field begins to oscillate about its vacuum state at the minimum of the potential. Then, the energy of the inflation field must decay into the standard model of particles, and this process is called as reheating, which is beyond the scope of this discussion.

4.3. Using EoS

We saw that simply the vacuum energy (Λ) with the EoS $p = -\rho$ gives the desired exponential expansion; however, the mechanism behind the inflation cannot be a cosmological constant simply because a Universe dominated by the vacuum energy stays dominated by it for the infinite future, then a radiation dominated era will never be reached, so the “graceful exit” is not possible in this scenario. But, the desired expansion can be reached by a simple modification on the EoS as

$$p = -\rho + f(\rho) . \tag{50}$$

The scalar field models can be reconstructed with a perfect fluid. There is also an increasing interest to imperfect fluids in the scientific community in order to explain the inflation better and to unify inflation with dark energy. For this purpose, two types of fluids have recently received attention, namely the Chaplygin gas[9][10][11][12] and viscous fluids[13][14][15][16]. For example the EoS of the modified Chaplygin gas can be given by

$$p = A\rho - \frac{B}{\rho^{\alpha}} \,,$$

where A, B, α are the free parameters needed to be determined by observations. The viscosity term can be added to the EoS as a function of Hubble parameter H and its derivatives as

$$p = -\rho + f(H, \dot{H}, \ddot{H}, \dots) \,.$$

In these models the slow-roll and observable parameters can also be obtainable but here, the details of this formalism will not be discussed further.

4.4. Modified Gravity Theories

Up to now, we have discussed how to realize the inflation by adding a hypothetical field or fluid to the RHS of Einstein's equation (2). The another route to make this happen is to modify the Einstein's relativity, namely the LHS of the equation. Since the inflation is thought to be happened at the very early stage of the Universe, the energy scale is very high, and at this scale the effective theory could be slightly different than the General Relativiy. It is possible that the quantum gravitational effects could play crutial role at this level, so people are investigating other theories to find out if the inflation is possible in those theories. Then, the general route to modify the Einstein's gravity is adding some geometric terms to the Einstein-Hilbert action. The Einstein-Hilbert action is given by

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{g}R + S_m \,, \tag{51}$$

where R is the Ricci scalar and S_m is the matter action. The generalization of this action, on the other hand, can be done by replacing R with a function of curvature terms as

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{g}f(R, R_{\mu\nu}, R_{\mu\nu\rho\sigma}, G, T, ?) + S_m \,. \tag{52}$$

Then, varying this action with respect to the metric gives the equations of motion similar to the Einstein's equation.

One of the examples of this kind is R^2 gravity where the function f can be given by $f(R) = R + \alpha R^2$. The inflation in this theory is called the Starobinsky inflation, which is the earliest inflationary model proposed by Alexei Starobinsky in 1980[17]. In this model, the modification to the action comes from some complicated renormalization procedures of conformal fields, but it result in graceful exit free inflation model with quasi-de Sitter expansion. There are so many model in this route and inflation in modified gravity theories is still an active research area.

5. Conclusion

In this report, I try to introduce the inflationary cosmology. To conclude, it can be said that the inflation is a paradigm to solve the problems of old big bang theory, some of which are presented in this report. Starting with the Guth's paper, it has taken a large attention in the science community and today, it is widely accepted idea but needs strong observational supports. Nevertheless, since the mechanism behind it is still not understood well, the inflation theory is one of the main branch in the studies of theoretical cosmology.

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