ME 306 Fluid Mechanics II

Part 1

Potential Flow

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Inviscid (Frictionless) Flow

- Continuity and Navier-Stokes equations govern the flow of fluids.
- For incompressible flows of Newtonian fluids they are

$$\nabla \cdot \vec{V} = 0$$

$$\rho \frac{D\vec{V}}{Dt} = \rho \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} \right] = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{V}$$

- These equations can be solved analytically only for a few problems.
- They can be simplified in various ways.
- Common fluids such as water and air have small viscosities. $\mu_{water} = 10^{-3}$ Pa s, $\mu_{air} = 2 \times 10^{-5}$ Pa s
- Neglecting the viscous term (zero shear force) gives the Euler's equation.

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p$$

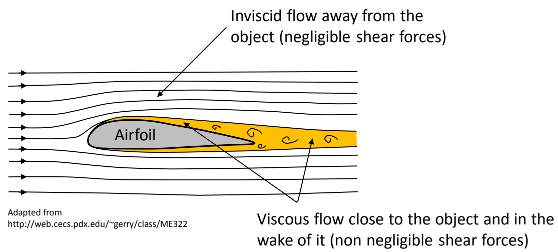
- Still difficult get a general analytical solution for the unknowns p and $\vec{V}$.

Inviscid Flow (cont'd)

- Bernoulli Equation (BE) is Euler's equation written along a streamline.

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = \text{constant along a streamline}$$

Exercise: Starting from the Euler's equation derive the BE.



Adapted from http://web.cecs.pdx.edu/~gerry/class/ME322

Inviscid and Irrotational Flow

- To simplify further we can assume the flow to be irrotational.

$$\vec{\xi} = 2\vec{\omega} = \nabla \times \vec{V} = 0 \quad \text{(irrotational flow)}$$

Vorticity (ksi)   Angular velocity

- Question: What's the logic behind irrotationality assumption?
- Irrotationality is about velocity gradients.

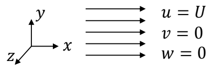
$$\vec{\xi} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \vec{k} = 0$$

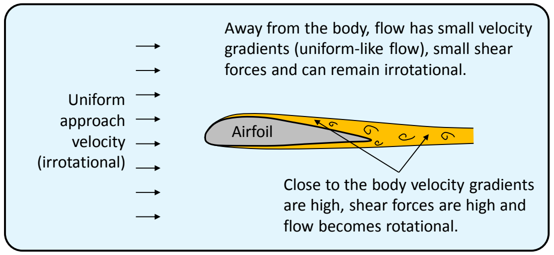
or

$$\vec{\xi} = \left(\frac{1}{r} \frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z} \right) \vec{i_r} + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} \right) \vec{i_\theta} + \left(\frac{1}{r} \frac{\partial (r V_\theta)}{\partial r} - \frac{1}{r} \frac{\partial V_r}{\partial \theta} \right) \vec{i_z}$$

Inviscid and Irrotational Flow (cont'd)

- One special irrotational flow is when all velocity gradients are zero.
- An example is uniform flow such as $u = U$, $v = 0$, $w = 0$.
- In many flow fields there will be uniform-like flow regions.

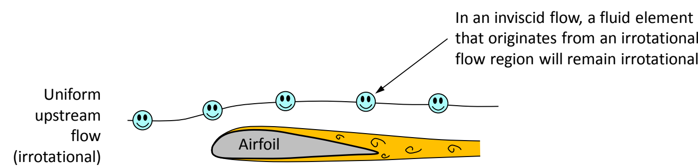




Exercise: Sketch the developing flow inside a pipe with uniform entrance and show the uniform and non-uniform flow regions.

Inviscid and Irrotational Flow (cont'd)

- In an inviscid flow net shear force acting on a fluid element is zero.
- Only pressure and body forces act on the fluid element. But they cannot cause rotation because
  - pressure forces act perpendicular to the element's surface.
  - body forces act through element's center of gravity.



Exercise: Show how a fluid element will rotate inside the developing flow region of a pipe with uniform entrance.

Inviscid and Irrotational Flow (cont'd)

- In general, flow fields are composed of both
  - irrotational regions with negligible shear forces
  - and rotational regions with considerable shear forces

Note: There are other factors that can cause rotation, but they are not as common as viscous effects.

- Sometimes rotational regions will be very thin such as high speed external flow over an airfoil.
- But still neglecting them and assuming the flow to be totally irrotational would yield unrealistic results.

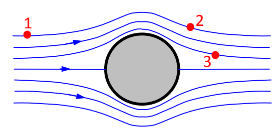
Assuming external flow over a body to be inviscid and irrotational everywhere will result in zero air drag, which is not correct. This is known as d'Alambert's paradox.

Exercise: What will happen if we assume pipe flow with uniform entrance to be inviscid and irrotational?

BE for Irrotational Flow

Exercise: Repeat the exercise of Slide 1-3 (derive BE) for irrotational flow.

- In an irrotational flow BE is valid between any two points of the flow field, not necessarily two points on the same streamline.



Inviscid, irrotational flow over an object

$$\left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_1 = \left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_2 = \left(\frac{p}{\rho g} + \frac{V^2}{2g} + z\right)_3$$

Velocity Potential (ϕ)

- For an irrotational flow : $\nabla \times \vec{V} = 0$
- As studied in ME 210, curl of the gradient of any scalar function is zero.

$$\nabla \times (\nabla \phi) = 0$$

- Using these two equations we can define a velocity potential function (ϕ) as

$$\vec{V} = +\nabla \phi$$

Some books use a minus sign so that ϕ decreases in the flow direction, similar to temperature decreasing in the heat flow direction. But we use plus in this course.

Phi: A scalar function called velocity potential

- In an irrotational flow field, velocity vector can be expressed as the gradient of a scalar function called the velocity potential.

Potential Flow

- For a 2D flow in the xy plane : $\vec{V} = \nabla \phi \quad \rightarrow \quad u = \frac{\partial \phi}{\partial x} \quad , \quad v = \frac{\partial \phi}{\partial y}$
- For a 2D flow in the $r\theta$ plane : $\vec{V} = \nabla \phi \quad \rightarrow \quad V_r = \frac{\partial \phi}{\partial r} \quad , \quad V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$

- If the irrotational flow is also incompressible (In ME 306 we'll NOT study compressible irrotational flows)

Continuity Equation : $\nabla \cdot \vec{V} = 0$

$$\nabla \cdot \nabla \phi = 0$$

$$\nabla^2 \phi = 0 \longleftarrow \text{Laplace's equation}$$

$\nabla^2 = \nabla \cdot \nabla$: Laplace operator

- For an incompressible and irrotational flow, velocity potential satisfies the Laplace's equation. These flows are called potential flows.

Velocity Potential (cont'd)

- $\nabla^2 \phi = 0$
  - In the xy plane : $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$
  - In the $r\theta$ plane : $\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$

Streamfunction

- Note that the relation between $\vec{V}$ and ϕ is similar to that of $\vec{V}$ and ψ.

Cauchy Riemann Equations

In the xy plane

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

In the $r\theta$ plane

$$V_r = \frac{\partial \phi}{\partial r} \qquad V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad V_\theta = -\frac{\partial \psi}{\partial r}$$

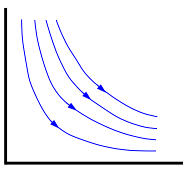
Potential Flow Exercises

Exercise : Using Cauchy Riemann equations show that streamfunction also satisfies the Laplace's equation for incompressible, potential flows.

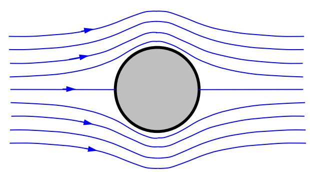
Exercise : Show that constant streamfunction lines (streamlines) are always perpendicular to constant velocity potential lines for incompressible, potential flows.

Exercise : Draw constant velocity potential lines of the following flow fields for which streamlines are shown. Constant velocity potential lines and streamlines drawn together form a flow net. What's the "heat transfer" analogue of a flow net?

Flow near a corner



Flow over a cylinder



Potential Flow Exercises (cont'd)

Exercise : The two-dimensional flow of a nonviscous, incompressible fluid in the vicinity of a corner is described by the stream function

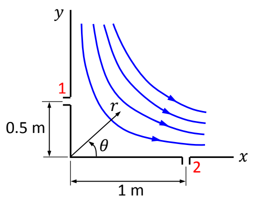
$$\psi = 2r^2 \sin(2\theta)$$

where ψ has units of m²/s when r is in meters. Assume the fluid density is 1000 kg/m³ and the xy plane is horizontal.

a) Determine, if possible, the corresponding velocity potential.

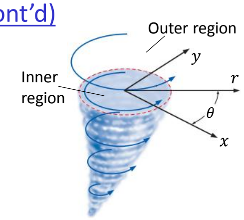
b) If the pressure at point 1 on the wall is 30 kPa, what is the pressure at point 2?

Reference: Munson's book.



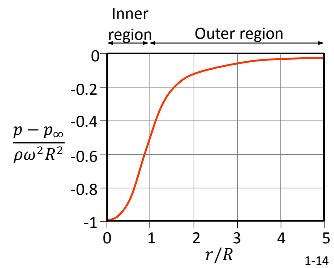
Potential Flow Exercises (cont'd)

Exercise : A horizontal slice through a tornado is modeled by two distinct regions. The inner or core region ($0 < r < R$) is modeled by solid body rotation. The outer region ($r > R$) is modeled as an irrotational region of flow. The flow is 2D in the $r\theta$-plane, and the components of the velocity field are given by



$$V_r = 0 \quad , \quad V_\theta = \begin{cases} \omega r & 0 < r < R \\ \dfrac{\omega R^2}{r} & r > R \end{cases}$$

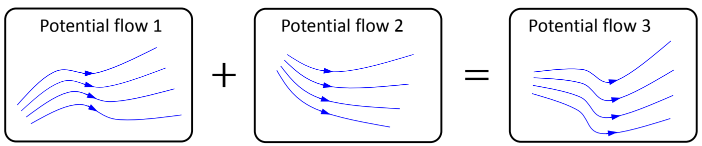
where ω is the magnitude of the angular velocity in the inner region. The ambient pressure (far away from the tornado) is equal to p_∞. Obtain the shown nondimensional pressure distribution.



Reference: Çengel's book.

Superposition of Elementary Potential Flows

- Laplace's equation is a linear PDE.
- Superposition can be applied to both velocity potential and streamfunction.



- $\phi_1 + \phi_2 = \phi_3 \quad , \quad \psi_1 + \psi_2 = \psi_3 \quad , \quad \vec{V}_1 + \vec{V}_2 = \vec{V}_3$
- To obtain complicated flow fields we can combine elementary ones such as
  - Uniform flow
  - Line source/sink
  - Vortex

1. Uniform Flow

- Consider uniform flow in the xy plane in $+x$ direction.

$$u = U \quad , \quad v = 0$$

- Let's find the equation for velocity potential.

$$u = \frac{\partial \phi}{\partial x} \quad \rightarrow \quad U = \frac{\partial \phi}{\partial x} \quad \rightarrow \quad \phi = Ux + f(y)$$

$$v = \frac{\partial \phi}{\partial y} \quad \rightarrow \quad 0 = \frac{\partial \phi}{\partial y} \quad \rightarrow \quad \frac{df}{dy} = 0 \quad \rightarrow \quad f = \text{constant}$$

- Taking $f = 0$ for simplicity

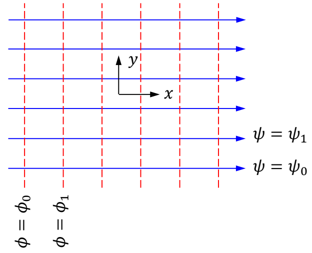
$$\phi = Ux$$

- Constant ϕ lines correspond to constant x lines, i.e. lines parallel to the y axis.

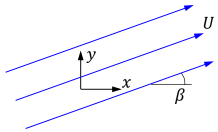
Exercise : Show that streamfunction equation is $\psi = Uy$

1. Uniform Flow (cont'd)

- Constant ϕ and constant ψ lines are shown below.

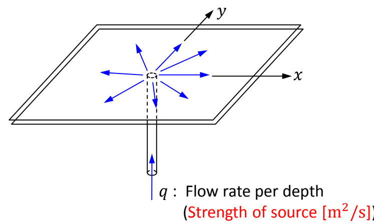


Exercise : Determine the equations of ϕ and ψ for uniform flow in a direction making an angle of β with the x axis.

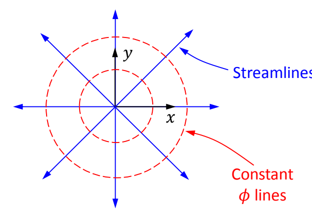


2. Line Source at the Origin

- Consider the 2D flow emerging at the origin of the xy plane and going radially outward in all directions with a total flow rate per depth of q.



View from the top



- Conservation of mass: $q = (2\pi r)V_r \quad \rightarrow \quad V_r = \dfrac{q}{2\pi r} \quad , \quad V_\theta = 0$
- V_r decreases with r, i.e. effect of source diminishes with r.
- Origin is a singular point with $V_r \rightarrow \infty$, which is not physical, so don't get too close.

2. Line Source (cont'd)

- Let's find the equation for velocity potential.

$$V_r = \frac{\partial \phi}{\partial r} \quad \rightarrow \quad \frac{q}{2\pi r} = \frac{\partial \phi}{\partial r} \quad \rightarrow \quad \phi = \frac{q}{2\pi}\ln(r) + f(\theta)$$

$$V_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \quad \rightarrow \quad 0 = \frac{1}{r}\frac{df}{d\theta} \quad \rightarrow \quad \frac{df}{d\theta} = 0 \quad \rightarrow \quad f = \text{constant}$$

- Taking $f = 0$ for simplicity

$$\phi = \frac{q}{2\pi}\ln(r)$$

- Constant ϕ lines correspond to constant r lines as shown in the previous slide.

Exercise : Show that the streamfunction equation is $\psi = \frac{q}{2\pi}\theta$

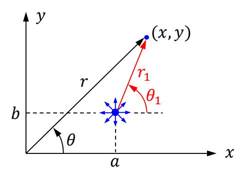
- To study a line sink for which the flow is radially inward towards a point, simply use a negative q value.

2. Line Source (cont'd)

- Consider a line source that is located NOT at the origin.
- Equations for ϕ and ψ change as follows

$$\phi = \frac{q}{2\pi}\ln(r_1)$$

$$\psi = \frac{q}{2\pi}\theta_1$$



or equivalently using x and y coordinates

$$\phi = \frac{q}{2\pi}\ln\left(\sqrt{(x-a)^2 + (y-b)^2}\right)$$

$$\psi = \frac{q}{2\pi}\arctan\left(\frac{y-b}{x-a}\right)$$

Some useful relations

$$x = r\cos(\theta)$$

$$y = r\sin(\theta)$$

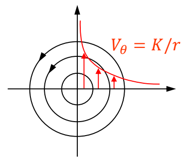
$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

3. Irrotational Vortex

- Studied in ME 305 as free vortex. Its velocity components are

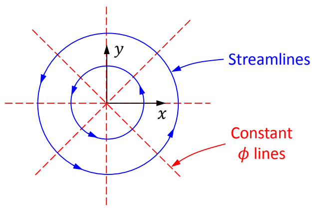
$$V_\theta = \frac{K}{r} \quad , \quad V_r = 0$$



- Using Cauchy Riemann relations we get

$$\phi = K\theta \quad , \quad \psi = -K\ln(r)$$

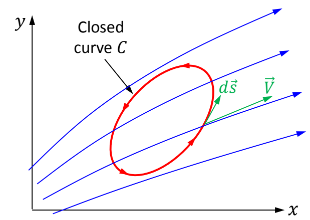
- Compared to a line source, streamlines and constant potential lines are interchanged.



Similar to a line source/sink, origin is a singular point, where the velocity shoots to infinity.

3. Irrotational Vortex (cont'd)

- Strength of a vortex is not given by K. Instead we use its circulation Γ.
- Circulation is the line integral of the tangential component of the velocity vector around a closed curve. It is related to the rotationality of the flow.



$$\Gamma = \oint_C \vec{V} \cdot d\vec{s} \qquad [\mathrm{m^2/s}]$$

Closed path of integration

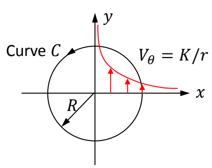
Differential vector along the path of integration

- For the 2D flow in the xy plane shown above

$$\left.\begin{aligned} \vec{V} &= u\vec{\imath} + v\vec{\jmath} \\ d\vec{s} &= dx\vec{\imath} + dy\vec{\jmath} \end{aligned}\right\} \quad \Gamma = \oint_C (udx + vdy)$$

3. Irrotational Vortex (cont'd)

Exercise: Calculate the circulation of an irrotational vortex for the following curve C



- Irrotational vortex is irrotational everywhere except the origin. All the circulation is squeezed into the origin, which is a singular point.
- Circulation $\Gamma = 2\pi K$ can be understood as the strength of the vortex. It's in $\mathrm{m^2/s}$.

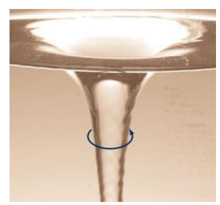
$$\phi = \frac{\Gamma}{2\pi}\theta \quad , \quad \psi = -\frac{\Gamma}{2\pi}\ln(r)$$

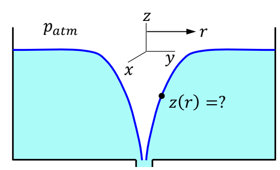
- Direction of the vortex is determined as $\Gamma > 0$: CCW ($+z$) rotating vortex
  $\Gamma < 0$: CW ($-z$) rotating vortex

3. Irrotational Vortex (cont'd)

Exercise: A liquid drains from a large tank through a small opening as illustrated. A vortex forms, whose velocity distribution away from the opening can be approximated as that of a free vortex. Determine an expression relating the surface shape to the strength of the vortex Γ.

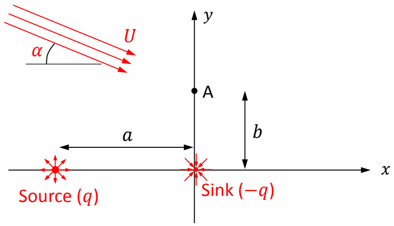
Reference: Munson's book





Exercises for Elementary Potential Flows

Exercise : Elementary components of a potential flow of water is shown below. Find the velocity and pressure at point A if the pressure at infinity is 100 kPa.



$U = 3$ m/s

$\alpha = 30°$

$q = 10\ \mathrm{m^2/s}$

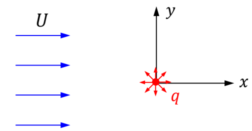
$a = 0.8$ m

$b = 0.6$ m

Exercise : For the previous problem determine the equations of velocity potential and streamfunction by superimposing elementary flows. Find the velocity at point A by differentiating both ϕ and ψ equation.

Source in a Uniform Flow (Flow Past a Half Body)

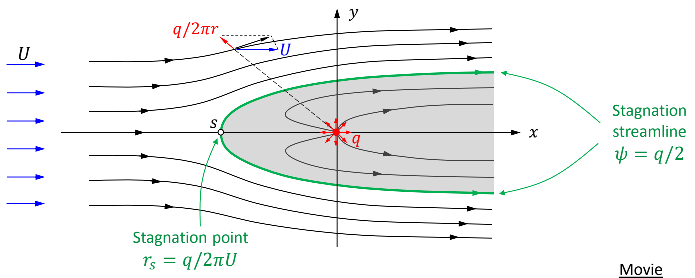
Exercise: Study the flow obtained by the combination of uniform flow in x direction and a source at the origin. Obtain the location of the stagnation point(s) and draw the stagnation streamline.



Uniform flow: $u = U$, $v = 0$, $\phi = Ux$, $\psi = Uy$

Source: $V_r = \frac{q}{2\pi r}$, $V_\theta = 0$, $\phi = \frac{q}{2\pi}\ln(r)$, $\psi = \frac{q}{2\pi}\theta$

Flow Past a Half Body (cont'd)



Movie
Flow Over Half Body

- Flow outside the stagnation streamline resembles a flow over a body with a blunt nose.
- Equation of the half body is given by the equation of the stagnation streamline.

Flow Past a Half Body (cont'd)

Exercise : Consider the top part of a half body. Draw speed vs. θ and pressure vs. θ using the following values: $\rho = 1000\ \mathrm{kg/m^3}$, $U = 5$ m/s , $q = 10\ \mathrm{m^2/s}$ and $p_\infty = 100$ kPa.

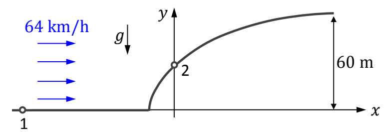
Exercise: A 64 km/h wind blows toward a hill that can be approximated with the top part of a half body. The maximum height of the hill approaches 60 m.

a) What is the air speed at a point directly above the origin (at point 2)?

b) What is the elevation of point 2?

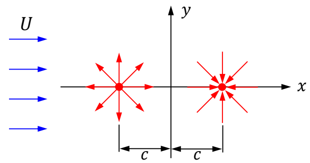
c) What is the pressure difference between point 2 and point 1 far from the hill?

Reference: Munson's book

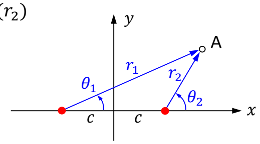


A Source and a Sink in Uniform Flow (Flow Past a Rankine oval)

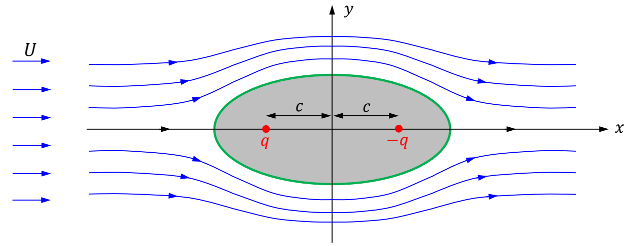
- Superposition of
  - a source of strength q at $x = -c$,
  - a sink of strength $-q$ at $x = c$ and
  - uniform flow of magnitude U.



- $\phi = \phi_{uni} + \phi_{sou} + \phi_{sink} = Ux + \frac{q}{2\pi}\ln(r_1) - \frac{q}{2\pi}\ln(r_2)$
- $\psi = \psi_{uni} + \psi_{sou} + \psi_{sink} = Uy + \frac{q}{2\pi}\theta_1 - \frac{q}{2\pi}\theta_2$



Flow Past a Rankine oval (cont'd)



Exercise : Determine the location of the stagnation points of the shown Rankine oval. Determine its length and thickness of the oval. Plot the variation of speed and pressure (with respect to p_∞) on it as a function of θ.

Doublet

- Superposition of
  - a source of strength q at the orgin (moved from $-x$ axis to the origin),
  - a sink of strength $-q$ at the origin (moved from $+x$ axis to the origin) ,
- Consider the limiting case of the source and sink of Slide 1-29 approaching to the origin. Skipping the details we can get

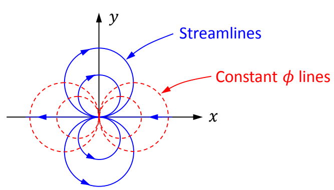
$$\phi_{doublet} = \frac{d}{2\pi r}\cos(\theta) \quad , \quad \psi_{doublet} = -\frac{d}{2\pi r}\sin(\theta)$$

where d is the strength of the doublet.

- Velocity field is given by

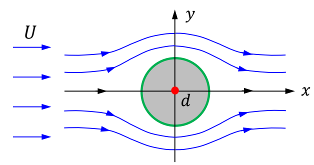
$$V_r = \frac{\partial\phi}{\partial r} = -\frac{d}{2\pi r^2}\cos(\theta)$$

$$V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{d}{2\pi r^2}\sin(\theta)$$



A Doublet in Uniform Flow (Flow Past a Cylinder)

- Superposition of
  - a doublet of strength d at the origin.
  - uniform flow of magnitude U in $+x$ direction.



- $\phi = \phi_{uni} + \phi_{doublet} = Ux + \frac{d}{2\pi r}\cos(\theta)$
- $\psi = \psi_{uni} + \psi_{doublet} = Uy - \frac{d}{2\pi r}\sin(\theta)$

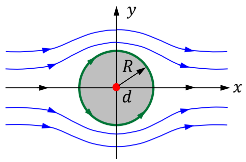
Flow Past a Cylinder (cont'd)

Important results are as follows

- Velocity components are

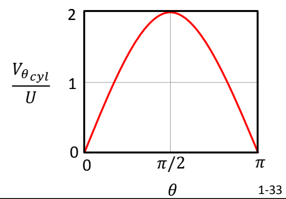
$$V_r = U\left(1-\frac{R^2}{r^2}\right)\cos(\theta)\ , \qquad V_\theta = -U\left(1+\frac{R^2}{r^2}\right)\sin(\theta)$$

with $R=\sqrt{d/U}$



- Stagnation points are located at $(-R,0)$ and $(R,0)$.
- Stagnation streamline is a circle of radius R.
- Velocity distribution on the cylinder is

$$V_{\theta\,cyl} = -2U\sin(\theta)$$

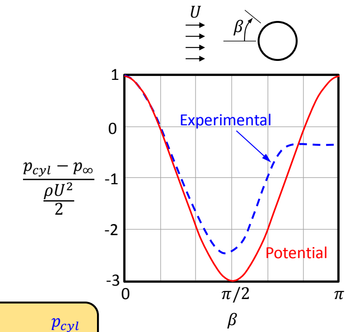


Flow Past a Cylinder (cont'd)

- Pressure distribution on the cylinder is (using BE with $V_\infty = U$ and p_∞)

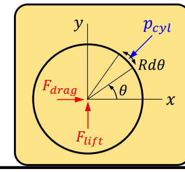
$$p_{cyl} = p_\infty + \frac{\rho U^2}{2}\left(1-4\sin^2(\theta)\right)$$

- Pressure on the cylinder is symmetric with respect to both x and y axis.
- Pressure does not create any drag force (in x direction) or any lift force (in y direction).



$$F_{drag} = -\int_0^{2\pi} p_{\mathrm{cyl}}\cos(\theta)R d\theta \ = 0$$

$$F_{lift} = -\int_0^{2\pi} p_{\mathrm{cyl}}\sin(\theta)R d\theta \ = 0$$



Flow Past a Cylinder (cont'd)

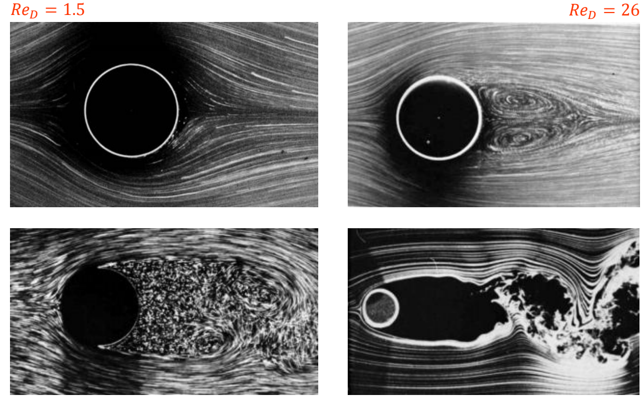
- As seen from the above exercise potential flow theory predicts ZERO DRAG FORCE on the cylinder.
- Actually this is the case for any closed body, irrespective of its shape.
- This result is not physical and it is known as d'Alembert paradox (1752).
- In a real viscous flow
  - shear stresses inside the boundary layer will cause a frictional drag force.
  - viscous action will cause separation & the pressure at the front and back of the cylinder would not be symmetric.

Movie
Flow with Separation

Movie
Potential vs. Viscous

Flow Past a Cylinder (cont'd)

$Re_D = 1.5$ $\qquad$ $Re_D = 26$



$Re_D = 2000$ $\qquad$ $Re_D = 10000$

"An Album of Fluid Motion", by M. Van Dyke

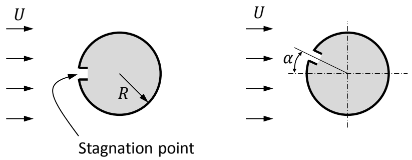
Flow Past a Cylinder (cont'd)

Exercise : When a small circular cylinder is placed in a uniform stream, a stagnation point is created on the cylinder. If a small hole is located at this point, the stagnation pressure, can be measured and used to determine the approach velocity, U (similar to a Pitot tube).

a) Show how p_{stag} and U are related. Pressure far away is p_∞.

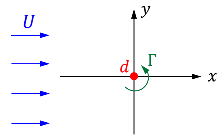
b) If the cylinder is misaligned by an angle α, but the measured pressure is still interpreted as the stagnation pressure, use potential flow theory to determine an expression for the ratio of the true velocity, U, to the predicted velocity, U'. Plot this ratio as a function of α for the range $0° < \alpha < 20°$.

Reference: Munson's book.



Flow Past a Rotating Cylinder

- Superposition of
  - a doublet of strength d located at the origin,
  - CCW rotating irrotational vortex of strength Γ located at the origin,
  - uniform flow of magnitude U.



- This will result in

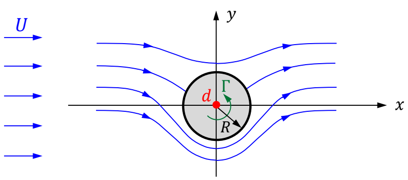
$$\phi = Ur\left(1+\frac{R^2}{r^2}\right)\cos(\theta)+\frac{\Gamma}{2\pi}\theta$$

$$\psi = Ur\left(1-\frac{R^2}{r^2}\right)\sin(\theta)-\frac{\Gamma}{2\pi}\ln(r)$$

with $R = \sqrt{d/U}$

- This is the potential flow that resembles the flow over a rotating cylinder of radius R.

Flow Past a Rotating Cylinder (cont'd)



Exercise : For the flow shown above, obtain the following results

$$V_{\theta\,cyl} = -2U\sin(\theta)+\frac{\Gamma}{2\pi R}$$

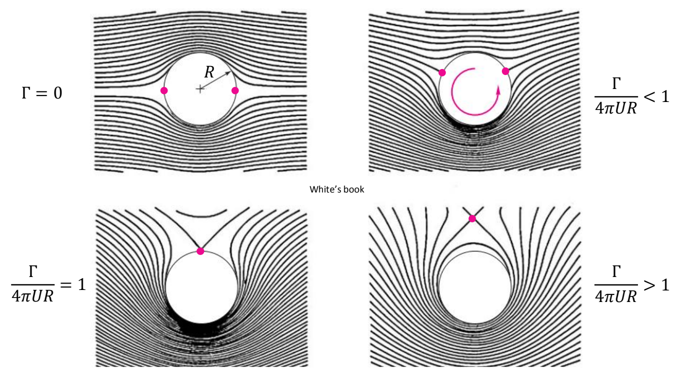
$$p_{cyl} = p_\infty + \frac{\rho U^2}{2}\left[1-4\sin^2(\theta)+\frac{2\Gamma}{\pi U R}\sin(\theta)-\left(\frac{\Gamma}{2\pi U R}\right)^2\right]$$

Integrate the above pressure distribution to get the following results

$$F_{drag} = 0 \quad , \quad \boxed{F_{lift} = -\rho\Gamma U} \text{ (per unit depth)}$$

Flow Past a Rotating Cylinder (cont'd)

- Streamlines and stagnation points for different circulation values.



Magnus Effect

- Magnus Effect: A rotating body in a uniform flow will have a net lift force on it (1853).
- Direction of the lift force depends on the direction of U and Γ.

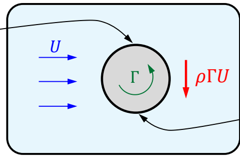
Top

$$V_{\theta\,top} = -2U + \frac{\Gamma}{2\pi R}$$

Opposite signs

Velocity is low.

Pressure is high.



Bottom

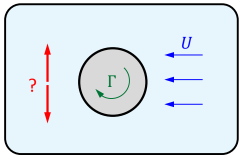
$$V_{\theta\,bottom} = 2U + \frac{\Gamma}{2\pi R}$$

Same signs

Velocity is high.

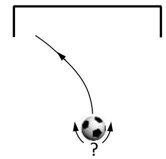
Pressure is low.

Exercise: Determine the direction of the lift force.



Magnus Effect (cont'd)

Exercise : Magnus effect acts not only on cylinders but also on other rotating bodies such as spheres. It can be used to explain how a spinning ball moves in a curved trajectory. A football player wants to make a penalty kick as sketched below. Will a CW or a CCW spin do the trick?



Exercise : Watch the following movies

https://www.youtube.com/watch?v=2OSrvzNW9FE (Suprising applications of Magnus effect)
http://www.youtube.com/watch?v=23f1jvGUWJs (Magnus force on Veritasium channel)
http://www.youtube.com/watch?v=2pQga7jxAyc (Enercon's rotor ship. Audio in German)
http://www.youtube.com/watch?v=wb5tc_nnMUw (Roberto Carlos knows the Magnus force)

Magnus Effect (cont'd)

- Warning: Although potential flow theory can predict the direction of lift force due to Magnus effect correctly, it may give quite inaccurate values for its magnitude. We'll come back to this in the next chapter.

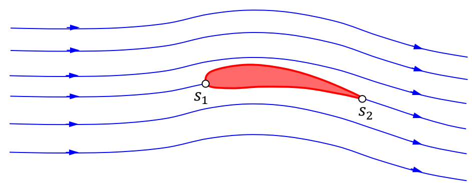
Exercise: In 1920s Anton Flettner built a series of rotor ships that are propelled by rotating cylinders driven by electric motors. Read about Flettner's ship at rexresearch.com/flettner/flettner.htm and understand how it works.

One of Flettner's original rotating cylinder ships

https://en.wikipedia.org/wiki/E-Ship_1

Kutta Condition (Lift on an Airfoil)

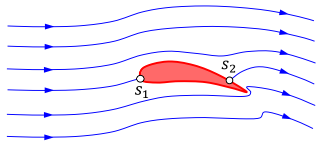
- Magnus effect applies not only to cylinders but any closed shape.
- Consider the flow over a slender body with a sharp trailing edge, such as an airfoil.
- An airfoil is designed to generate small drag and high lift force.



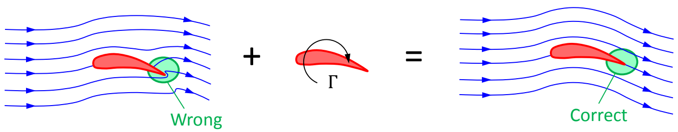
- There are two stagnation points, s_1 and s_2.
- Experiments show that the streamlines leave the trailing edge smoothly as shown above, known as the Kutta condition.

Lift on an Airfoil (cont'd)

- Potential flow theory will predict an unphysical location for point s_2.
- It is impossible for streamlines to make such a sharp turn at the trailing edge.



- If we add the correct amount of CW vortex to this flow field we can bring point s_2 down to the trailing edge and obtain the correct streamline pattern.

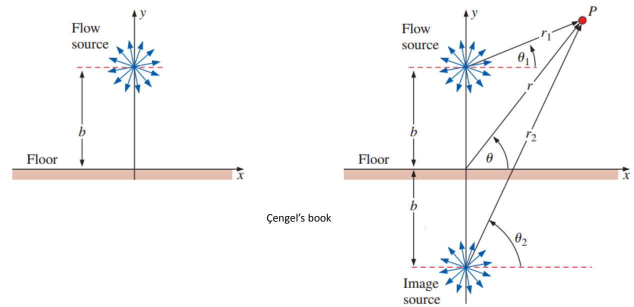


- The magnitude, Γ, of the necessary vortex can be used to calculate the lift force generated on the airfoil.

$$F_{lift} = \rho \, \Gamma \, U \uparrow$$ Kutta-Joukowski Law (1902)

Simulating Flows Near Walls using Mirror Images

- Consider a line source located at a distance b to a solid wall.
- Fluid cannot pass across the wall and therefore it is a streamline.
- The effect of the wall can be simulated by using another source, which is the mirror image of the original one with respect to the wall.



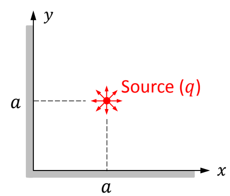
Simulating Flows Near Walls using Mirror Images (cont'd)

Exercise : Consider a source of strength q located close to two walls forming a 90° corner.

a) How many and where the mirror images need to be placed to simulate existence of the walls?

b) Locate the stagnation point(s).

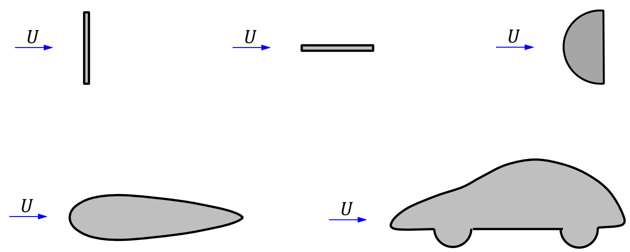
c) Draw the streamlines.



Exercise : Repeat the previous exercise by replacing the source with a clockwise vortex of strength Γ.

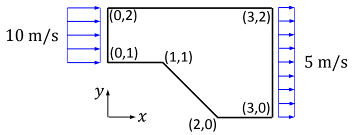
Superposition Exercises

Exercise : We want to study the potential flow over the following bodies. If possible, which elementary flows need to be superimposed to get the desired shape?



Numerical Solution of Potential Flow

- Obtaining complicated flow fields by superposing elementary ones is limited.
- To study potential flows on arbitrary geometries one can perform a numerical solution.
- Consider a flow inside an expanding duct (coordinates are in meters).



- Potential flow inside the duct can be obtained by solving Laplace's equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \quad , \qquad \text{with proper boundary conditions}$$

Numerical Solution of Potential Flow (cont'd)

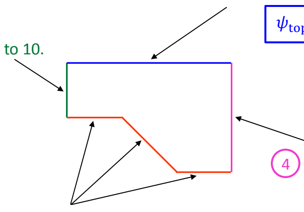
- Boundary conditions are (study in the given order)

2 Top wall is a streamline. ψ should be constant there. In order to have 10 m²/s flow rate per depth between the top and bottom walls

$\boxed{\psi_{\text{top}} = 10}$

3 At the inlet $u = 10$.
Therefore $\frac{\partial\psi}{\partial y} = 10$.
ψ varies linearly from 0 to 10.

$\boxed{\psi_{\text{left}} = 10y - 10}$



4 At the exit $u = 5$.
Therefore $\frac{\partial\psi}{\partial y} = 5$.
ψ varies linearly from 0 to 10.

$\boxed{\psi_{\text{right}} = 5y}$

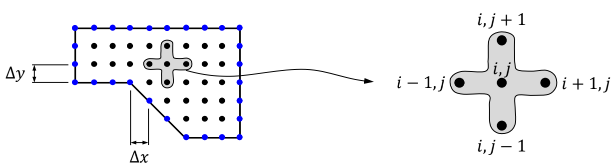
1 Bottom wall is a streamline. ψ should be constant there. Let's set it to zero.

$\boxed{\psi_{\text{bottom}} = 0}$

Numerical Solution of Potential Flow (cont'd)

- Finite Difference Method can be used to get the numerical solution.
- First we discretize the problem domain into a set of nodes.
- Following mesh has 55 nodes with $\Delta x = \Delta y = 1/3$.
- 27 of the nodes are at the boundaries and ψ is known at these nodes.
- 28 of the nodes are inside the domain and ψ needs to be calculated at them.

5 point computational stencil



Numerical Solution of Potential Flow (cont'd)

- Discretized form of the Laplace's equation at node (i,j) is

$$\underbrace{\frac{\psi_{i+1,j}-2\psi_{i,j}+\psi_{i-1,j}}{(\Delta x)^2}}_{\left.\frac{\partial^2\psi}{\partial x^2}\right|_{i,j}} + \underbrace{\frac{\psi_{i,j+1}-2\psi_{i,j}+\psi_{i,j-1}}{(\Delta y)^2}}_{\left.\frac{\partial^2\psi}{\partial y^2}\right|_{i,j}} = 0$$

- For $\Delta x = \Delta y$, discretized equation for node (i,j) becomes

$$\boxed{\psi_{i+1,j}+\psi_{i-1,j}+\psi_{i,j+1}+\psi_{i,j-1}-4\psi_{i,j}=0}$$

- This equation needs to be written for all non-boundary nodes.
- For nodes with boundary neighbors, some ψ values are known and they need to be transferred to the right-hand-side of the equation.
- At the end we'll get a system of 28 equations for 28 unknowns and solve it.

Numerical Solution of Potential Flow (cont'd)

- Following solution is obtained using a mesh with $\Delta x = \Delta y = 0.2$.

ψ =	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00
	8.00	8.02	8.04	8.07	8.12	8.20	8.30	8.41	8.52	8.62	8.71	8.79	8.85	8.91	8.95	9.00
	6.00	6.03	6.06	6.12	6.22	6.37	6.58	6.82	7.05	7.26	7.44	7.59	7.71	7.82	7.91	8.00
	4.00	4.03	4.07	4.13	4.26	4.48	4.84	5.24	5.61	5.93	6.19	6.41	6.59	6.74	6.88	7.00
	2.00	2.02	2.05	2.09	2.20	2.44	3.08	3.69	4.22	4.65	5.00	5.28	5.50	5.69	5.85	6.00
ψ =	0.00	0.00	0.00	0.00	0.00	0.00	1.33	2.22	2.92	3.45	3.87	4.19	4.45	4.66	4.84	5.00
							0.00	1.00	1.77	2.37	2.83	3.18	3.45	3.66	3.84	4.00
								0.00	0.80	1.42	1.90	2.24	2.50	2.70	2.86	3.00
									0.00	0.63	1.09	1.40	1.61	1.77	1.89	2.00
										0.00	0.44	0.66	0.79	0.87	0.94	1.00
											0.00	0.00	0.00	0.00	0.00	0.00

Red ones are boundary values

Black ones are calculated

White's book

- After obtaining the ψ values at the nodes, velocity components can be obtained

$$u = \frac{\partial \psi}{\partial y} \quad \rightarrow \quad u_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y}$$

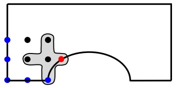
$$v = -\frac{\partial \psi}{\partial x} \quad \rightarrow \quad v_{i,j} = -\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x}$$

- Different formulas need to be used at the boundary nodes.

Numerical Solution of Potential Flow (cont'd)

- After obtaining u and v, pressures can be calculated using the Bernoulli equation.

- For irregular geometries modifications need to be done.



- Red node is NOT at Δx distance from the central node. What can be done?