Hypothesis Tests for μ

<table>
<tr><td rowspan="2">1</td><td>a
$H_0 : \mu = \mu_0$
$H_1 : \mu = \mu_1 \left(\mu_1 > \mu_0\right)$</td><td>b
$H_0 : \mu = \mu_0$
$H_1 : \mu > \mu_0$</td><td>c
$H_0 : \mu \le \mu_0$
$H_1 : \mu > \mu_0$</td></tr>
<tr><td colspan="3">σ^2 Known: Reject H_0 if $z_0 > z_\alpha$
σ^2 Unknown: Reject H_0 if $t_0 > t_{\alpha,n-1}$</td></tr>
<tr><td rowspan="2">2</td><td>$H_0 : \mu = \mu_0$
$H_1 : \mu = \mu_1 \left(\mu_1 < \mu_0\right)$</td><td>$H_0 : \mu = \mu_0$
$H_1 : \mu < \mu_0$</td><td>$H_0 : \mu \ge \mu_0$
$H_1 : \mu < \mu_0$</td></tr>
<tr><td colspan="3">σ^2 Known: Reject H_0 if $z_0 < -z_\alpha$
σ^2 Unknown: Reject H_0 if $t_0 < -t_{\alpha,n-1}$</td></tr>
<tr><td rowspan="2">3</td><td colspan="3">$H_0 : \mu = \mu_0$
$H_1 : \mu \ne \mu_0$</td></tr>
<tr><td colspan="3">σ^2 Known: Reject H_0 if $z_0 < -z_{\alpha/2}$ or $z_0 > z_{\alpha/2}$
σ^2 Unknown: Reject H_0 if $t_0 < -t_{\alpha/2,n-1}$ or $t_0 > t_{\alpha/2,n-1}$</td></tr>
</table>

σ^2 Known: $X \sim N\left(\mu,\sigma^2\right)$ $\quad n \ge 1,\ z = \dfrac{\overline{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1),\ z_0 = \dfrac{\overline{x}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$

1. and 2. a) MPT **3.** No UMPT (GLRT for exact test)
1. and 2. b), c) UMPT

X ~ Non-Normal, n large, $Z \overset{approximately}{\underset{CLT}{\sim}} N(0,1) \rightarrow$ same tests but results are approximate.

σ^2 Unknown: $\quad n \ge 1,\ \text{t} = \dfrac{\overline{x}-\mu}{s/\sqrt{n-1}} \sim t_{n-1},\ \text{t}_0 = \dfrac{\overline{x}-\mu_0}{s/\sqrt{n-1}} \sim t_{n-1}$

X ~ Non-Normal, n large, $z = \dfrac{\overline{x}-\mu}{s/\sqrt{n}} \overset{approx.}{\underset{CLT}{\sim}} N(0,1),\ z_0 = \dfrac{\overline{x}-\mu_0}{s/\sqrt{n}} \overset{approx.}{\underset{CLT}{\sim}} N(0,1)$

Hypothesis Tests for σ^2

	a	b	c
1	$H_0 : \sigma^2 = \sigma_0^2$ $H_1 : \sigma^2 = \sigma_1^2 \left(\sigma_1^2 > \sigma_0^2\right)$	$H_0 : \sigma^2 = \sigma_0^2$ $H_1 : \sigma^2 > \sigma_0^2$	$H_0 : \sigma^2 \le \sigma_0^2$ $H_1 : \sigma^2 > \sigma_0^2$
	μ Known: Reject H_0 if $\chi_0^2 > \chi_{\alpha,n}^2$ μ Unknown: Reject H_0 if $\chi_0^2 > \chi_{\alpha,n-1}^2$		
2	$H_0 : \sigma^2 = \sigma_0^2$ $H_1 : \sigma^2 = \sigma_1^2 \left(\sigma_1^2 < \sigma_0^2\right)$	$H_0 : \sigma^2 = \sigma_0^2$ $H_1 : \sigma^2 < \sigma_0^2$	$H_0 : \sigma^2 \ge \sigma_0^2$ $H_1 : \sigma^2 < \sigma_0^2$
	μ Known: Reject H_0 if $\chi_0^2 < \chi_{1-\alpha,n}^2$ μ Unknown: Reject H_0 if $\chi_0^2 < \chi_{1-\alpha,n-1}^2$		
3	$H_0 : \sigma^2 = \sigma_0^2$ $H_1 : \sigma^2 \ne \sigma_0^2$		
	μ Known: Reject H_0 if $\chi_0^2 < \chi_{1-\alpha/2,n}^2$ or $\chi_0^2 > \chi_{\alpha/2,n}^2$ μ Unknown: Reject H_0 if $\chi_0^2 < \chi_{1-\alpha/2,n-1}^2$ or $\chi_0^2 > \chi_{\alpha/2,n-1}^2$		

μ Known: $X \sim N\left(\mu,\sigma^2\right)$ $n \ge 1,\ \chi^2 = \dfrac{nS'^2}{\sigma^2} \sim \chi_n^2,\ \chi_0^2 = \dfrac{nS'^2}{\sigma_0^2} \overset{\sigma^2=\sigma_0^2}{\sim} \chi_n^2$

where $S'^2 = \dfrac{\sum_{i=1}^{n}\left(X_i - \mu\right)^2}{n}$.

1. and 2. a) MPT **3.** No UMPT (GLRT for exact test)

1. and 2. b), c) UMPT

X ~ Non-Normal: No general result.

μ Unknown: $X \sim N\left(\mu,\sigma^2\right)$ $n > 1,\ \chi^2 = \dfrac{nS^2}{\sigma^2} \sim \chi_{n-1}^2,\ \chi_0^2 = \dfrac{nS^2}{\sigma_0^2} \overset{\sigma^2=\sigma_0^2}{\sim} \chi_{n-1}^2$

X ~ Non-Normal: No general result.

Hypothesis Tests for Proportion, p

<table>
<tr><td rowspan="2">1</td><td>a
$H_0 : p = p_0$
$H_1 : p = p_1 \left(p_1 > p_0 \right)$</td><td>b
$H_0 : p = p_0$
$H_1 : p > p_0$</td><td>c
$H_0 : p \le p_0$
$H_1 : p > p_0$</td></tr>
<tr><td colspan="3">Exact Test: Reject H$_0$ if $y = \sum_{i=1}^{n} x_i \ge c$
$\alpha = P\left[Y \ge c \middle| p = p_0 \right] = P\left[Bin\left(n, p_0\right) \ge c \right]$ may require randomization
$\Pi\left(p\right) = P\left[Y \ge c \middle| p \right] = P\left[Bin\left(n, p\right) \ge c \middle| p \right]$
Approximate Test: Reject H$_0$ if $z_0 > z_\alpha$</td></tr>
<tr><td rowspan="2">2</td><td>$H_0 : p = p_0$
$H_1 : p = p_1 \left(p_1 < p_0 \right)$</td><td>$H_0 : p = p_0$
$H_1 : p < p_0$</td><td>$H_0 : p \ge p_0$
$H_1 : p < p_0$</td></tr>
<tr><td colspan="3">Exact Test: Reject H$_0$ if $y = \sum_{i=1}^{n} x_i \le c$
$\alpha = P\left[Y \le c \middle| p = p_0 \right] = P\left[Bin\left(n, p_0\right) \le c \right]$ may require randomization
$\Pi\left(p\right) = P\left[Y \le c \middle| p \right] = P\left[Bin\left(n, p\right) \le c \middle| p \right]$
Approximate Test: Reject H$_0$ if $z_0 < -z_\alpha$</td></tr>
<tr><td rowspan="2">3</td><td colspan="3">$H_0 : p = p_0$
$H_1 : p \ne p_0$</td></tr>
<tr><td colspan="3">Exact Test: Reject H$_0$ if $y = \sum_{i=1}^{n} x_i \ge c_1$ or $y = \sum_{i=1}^{n} x_i \le c_2$
$\alpha = P\left[Y \le c_1 \middle| p = p_0 \right] + P\left[Y \ge c_2 \middle| p = p_0 \right] = P\left[Bin\left(n, p_0\right) \le c_1 \right] + P\left[Bin\left(n, p_0\right) \ge c_2 \right]$
may require randomization
$\Pi\left(p\right) = P\left[Y \le c_1 \middle| p \right] + P\left[Y \ge c_2 \middle| p \right] = P\left[Bin\left(n, p\right) \le c_1 \right] + P\left[Bin\left(n, p\right) \ge c_2 \right]$
Approximate Test: Reject H$_0$ if $z_0 < -z_{\alpha/2}$ or $z_0 > z_{\alpha/2}$</td></tr>
</table>

Exact test: $X \sim Ber\left(p\right), Y = \sum_{i=1}^{n} X_i \sim Bin\left(n, p\right), E\left(Y\right) = np, V\left(Y\right) = np\left(1-p\right)$

1. and 2. a) MPT **3.** No UMPT (GLRT for exact test)

1. and 2. b), c) UMPT

Approximate Test:

large n, $Z = \dfrac{Y - np}{\sqrt{np\left(1-p\right)}} = \dfrac{\hat{p} - p}{\sqrt{\dfrac{p\left(1-p\right)}{n}}} \overset{approx.}{\sim} N\left(0,1\right)$, $Z_0 = \dfrac{Y - np_0}{\sqrt{np_0\left(1-p_0\right)}} = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0\left(1-p_0\right)}{n}}} \overset{approx.}{\sim} N\left(0,1\right)$

Hypothesis Tests for $\mu_1 - \mu_2$

	a	b	c
1	$H_0 : \mu_1 - \mu_2 = d_0$ $H_1 : \mu_1 - \mu_2 = d_1 \left(d_1 > d_0 \right)$	$H_0 : \mu_1 - \mu_2 = d_0$ $H_1 : \mu_1 - \mu_2 > d_0$	$H_0 : \mu_1 - \mu_2 \le d_0$ $H_1 : \mu_1 - \mu_2 > d_0$
	σ^2 Known: Reject H$_0$ if $z_0 > z_\alpha$ σ^2 Unknown but $\sigma_1^2 = \sigma_2^2 = \sigma^2$: Reject H$_0$ if $t_0 > t_{\alpha, n_1+n_2-2}$		
2	$H_0 : \mu_1 - \mu_2 = d_0$ $H_1 : \mu_1 - \mu_2 = d_1 \left(d_1 < d_0 \right)$	$H_0 : \mu_1 - \mu_2 = d_0$ $H_1 : \mu_1 - \mu_2 < d_0$	$H_0 : \mu_1 - \mu_2 \ge d_0$ $H_1 : \mu_1 - \mu_2 < d_0$
	σ^2 Known: Reject H$_0$ if $z_0 < -z_\alpha$ σ^2 Unknown but $\sigma_1^2 = \sigma_2^2 = \sigma^2$: Reject H$_0$ if $t_0 < -t_{\alpha, n_1+n_2-2}$		
3	$H_0 : \mu_1 - \mu_2 = d_0$ $H_1 : \mu_1 - \mu_2 \ne d_0$		
	σ^2 Known: Reject H$_0$ if $z_0 < -z_{\alpha/2}$ or $z_0 > z_{\alpha/2}$ σ^2 Unknown but $\sigma_1^2 = \sigma_2^2 = \sigma^2$: Reject H$_0$ if $t_0 < -t_{\alpha/2, n_1+n_2-2}$ or $t_0 > t_{\alpha/2, n_1+n_2-2}$		

X and Y are independent from each other.

σ_1^2 and σ_2^2 are known:

$$X \sim N\left(\mu_1, \sigma_1^2\right),\ n_1 \ge 1,\ Y \sim N\left(\mu_2, \sigma_2^2\right),\ n_2 \ge 1,\ z = \frac{\left(\bar{X} - \bar{Y}\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N\left(0,1\right)$$

$$z_0 = \frac{\left(\bar{X} - \bar{Y}\right) - d_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \overset{H_0}{\sim} N\left(0,1\right)$$

1. and 2. a) MPT **3.** No UMPT (GLRT for exact test)
1. and 2. b), c) UMPT

σ_1^2 and σ_2^2 are unknown:

$X \sim N\left(\mu_1, \sigma_1^2\right)$ or non-normal, n_1 large, $Y \sim N\left(\mu_2, \sigma_2^2\right)$ or non-normal, n_2 large,

$$z = \frac{\left(\bar{X} - \bar{Y}\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \overset{approx.}{\sim} N\left(0,1\right),\ z_0 = \frac{\left(\bar{X} - \bar{Y}\right) - d_0}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \overset{approx.}{\sim} N\left(0,1\right)$$

σ_1^2 and σ_2^2 are unknown but $\sigma_1^2 = \sigma_2^2 = \sigma^2$:

$$X \sim N\left(\mu_1, \sigma_1^2\right)\ ,\ Y \sim N\left(\mu_2, \sigma_2^2\right)\ ,\ n_1 + n_2 > 2,$$

$$t = \frac{\left(\bar{X} - \bar{Y}\right) - \left(\mu_1 - \mu_2\right)}{S_p^2 \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}\, ,\ t_0 = \frac{\left(\bar{X} - \bar{Y}\right) - d_0}{S_p^2 \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}$$

where $\mathrm{S}_p^2 = \frac{n_1 S_1^2 + n_2 S_2^2}{n_1 + n_2 - 2}$

X~Non-normal and/or Y~Non-normal, large sample sizes:

$$\mathrm{z} = \frac{\left(\bar{X} - \bar{Y}\right) - \left(\mu_1 - \mu_2\right)}{S_p^2 \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \overset{approx}{\sim} N\left(0,1\right),\ \mathrm{z}_0 = \frac{\left(\bar{X} - \bar{Y}\right) - d_0}{S_p^2 \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \overset{approx.}{\sim} N\left(0,1\right)$$

where $\mathrm{S}_p^2 = \frac{n_1 S_1^2 + n_2 S_2^2}{n_1 + n_2 - 2}$

Hypothesis Tests for σ_1^2 / σ_2^2

	a	b	c
1	$H_0 : \sigma_1^2 / \sigma_2^2 = k_0$ $H_1 : \sigma_1^2 / \sigma_2^2 = k_1 \left(k_1 > k_0 \right)$	$H_0 : \sigma_1^2 / \sigma_2^2 = k_0$ $H_1 : \sigma_1^2 / \sigma_2^2 > k_0$	$H_0 : \sigma_1^2 / \sigma_2^2 \le k_0$ $H_1 : \sigma_1^2 / \sigma_2^2 > k_0$
	μ_1 and μ_2 Known: Reject H$_0$ if $F_0 < F_{1-\alpha}\left(n_2, n_1\right)$ μ_1 and μ_2 Unknown: Reject H$_0$ if $F_0 < F_{1-\alpha}\left(n_2 - 1, n_1 - 1\right)$		
2	$H_0 : \sigma_1^2 / \sigma_2^2 = k_0$ $H_1 : \sigma_1^2 / \sigma_2^2 = k_1 \left(k_1 < k_0 \right)$	$H_0 : \sigma_1^2 / \sigma_2^2 = k_0$ $H_1 : \sigma_1^2 / \sigma_2^2 < k_0$	$H_0 : \sigma_1^2 / \sigma_2^2 \ge k_0$ $H_1 : \sigma_1^2 / \sigma_2^2 < k_0$
	μ_1 and μ_2 σ^2 Known: Reject H$_0$ if $F_0 > F_{\alpha}\left(n_2, n_1\right)$ μ_1 and μ_2 σ^2 Unknown: Reject H$_0$ if $F_0 < F_{1-\alpha}\left(n_2 - 1, n_1 - 1\right)$		
3	$H_0 : \sigma_1^2 / \sigma_2^2 = k_0$ $H_1 : \sigma_1^2 / \sigma_2^2 \ne k_0$		
	μ_1 and μ_2 Known: Reject H$_0$ if $F_0 < F_{1-\alpha/2}\left(n_2, n_1\right)$ or $F_0 < F_{\alpha/2}\left(n_2, n_1\right)$ μ_1 and μ_2 Unknown: Reject H$_0$ if $F_0 < F_{1-\alpha/2}\left(n_2 - 1, n_1 - 1\right)$ or $F_0 < F_{\alpha/2}\left(n_2 - 1, n_1 - 1\right)$		

μ_1 and μ_2 known: X and Y are independent from each other.

$$X \sim N\left(\mu_1, \sigma_1^2\right),\ n_1 \ge 1,\ Y \sim N\left(\mu_2, \sigma_2^2\right),\ n_2 \ge 1,$$

$$\mathrm{F} = \frac{\sigma_1^2}{\sigma_2^2}\frac{S_2'^2}{S_1'^2} \sim F\left(n_2, n_1\right),\ \mathrm{F}_0 = k_0 \frac{S_2'^2}{S_1'^2} \overset{H_0}{\sim} F\left(n_2, n_1\right)$$

No general result for the non-normal case.

μ_1 and μ_2 unknown:

$$X \sim N\left(\mu_1, \sigma_1^2\right),\ n_1 > 1,\ Y \sim N\left(\mu_2, \sigma_2^2\right),\ n_2 > 1,$$

$$\mathrm{F} = \frac{\sigma_1^2}{\sigma_2^2}\frac{S*_2^2}{S*_1^2} \sim F\left(n_2 - 1, n_1 - 1\right),\ \mathrm{F}_0 = k_0 \frac{S*_2^2}{S*_1^2} \overset{H_0}{\sim} F\left(n_2 - 1, n_1 - 1\right)$$

$$\text{where } S*^2 = \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}{n-1}.$$

Hypothesis Tests for $p_1 - p_2$

	a	b	c
1	$H_0 : p_1 - p_2 = d_0$ $H_1 : p_1 - p_2 = d_1 (d_1 > d_0)$	$H_0 : p_1 - p_2 = d_0$ $H_1 : p_1 - p_2 > d_0$	$H_0 : p_1 - p_2 \le d_0$ $H_1 : p_1 - p_2 > d_0$
	Approximate Test for large n_1 and n_2: Reject H_0 if $z_0 > z_\alpha$		
2	$H_0 : p_1 - p_2 = d_0$ $H_1 : p_1 - p_2 = d_1 (d_1 < d_0)$	$H_0 : p_1 - p_2 = d_0$ $H_1 : p_1 - p_2 < d_0$	$H_0 : p_1 - p_2 \ge d_0$ $H_1 : p_1 - p_2 < d_0$
	Approximate Test for large n_1 and n_2: Reject H_0 if $z_0 < -z_\alpha$		
3	$H_0 : p_1 - p_2 = d_0$ $H_1 : p_1 - p_2 \ne d_0$		
	Approximate Test for large n_1 and n_2: Reject H_0 if $z_0 < -z_{\alpha/2}$ or $z_0 > z_{\alpha/2}$		

$$X_1 \sim Ber(p_1), Y_1 = \sum_{i=1}^{n} X_{1,i} \sim Bin(n_1, p_1), E(Y_1) = n_1 p_1, V(Y_1) = n_1 p_1 (1 - p_1)$$

$$X_2 \sim Ber(p_2), Y_2 = \sum_{i=1}^{n} X_{2,i} \sim Bin(n_2, p_2), E(Y_2) = n_2 p_2, V(Y_2) = n_2 p_2 (1 - p_2)$$

X_1 and X_2 are independent from each other.

$$E[\hat{p}_1 - \hat{p}_2] = p_1 - p_2 \text{ and } V[\hat{p}_1 - \hat{p}_2] = \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}.$$

$$Y_1 \underset{\text{large } n_1}{\overset{approx.}{\sim}} N[n_1 p_1, n_1 p_1 (1 - p_1)]$$

$$Z = \frac{(Y_1 - Y_2) - (n_1 p_1 - n_2 p_2)}{\sqrt{n_1 p_1 (1-p_1) + n_2 p_2 (1-p_2)}} = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}} \underset{\text{large } n_1, n_2}{\overset{approx.}{\sim}} N(0,1)$$

$$Z_0 = \frac{(\hat{p}_1 - \hat{p}_2) - d_0}{\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}} \underset{\text{large } n_1, n_2}{\overset{approx.}{\sim}} N(0,1)$$