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On the Eigenstructure of DFT Matrices

The discrete Fourier transform (DFT) not only enables fast implementation of the discrete convolution operation, which is critical for the efficient processing of analog signals through digital means, but it also represents a rich and beautiful analytical structure that is interesting on its own. A typical senior-level digital signal processing (DSP) course involves a fairly detailed treatment of DFT and a list of related topics, such as circular shift, correlation, convolution operations, and the connection of circular operations with the linear operations [1]. Despite having detailed expositions on DFT, most DSP textbooks (including advanced ones) lack discussions on the eigenstructure of the DFT matrix. Here, we present a self-contained exposition on such.

Our goals are to study the eigenvalues and eigenvectors of the DFT matrix, to determine the multiplicity of the eigenvalues, to define the invariant subspaces under DFT mapping, to construct the projectors to the invariant subspaces and to underline some connections between invariant subspaces and other transforms.

(We believe that this discussion can be followed by most of the signal processing community, including advanced undergraduate students. The concepts used in this discussion are mostly elementary and available in standard linear algebra textbooks. A comprehensive knowledge of linear spaces is not required but would be highly beneficial to fully interpret some of the results. These notes have been prepared as an assignment for supplementary reading material on the DFT.)

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DESCRIPTION OF THE PROBLEM

Let **F** be a $N \times N$ unitary DFT matrix:

$$[\mathbf{F}]_{k,n} = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}nk}.$$

In the equation above, $[\mathbf{F}]_{k,n}$ denotes the matrix entry in the kth row and nth column of the matrix **F**. We assume both k and n run from 0 to $N-1$, following the literature on the DFT.

THE EIGENVALUES OF A MATRIX ARE, BY DEFINITION, THE ROOTS OF ITS CHARACTERISTIC POLYNOMIAL.

Different from the conventional definition given in [1], the definition above includes a scaling factor of $1\sqrt{N}$. This factor is required to make the matrix **F** unitary. From the theory of matrices, we know that the unitary matrices satisfy the relation $\mathbf{F}^H\mathbf{F} = \mathbf{I}$(another form of Parseval's relation) and have unit norm eigenvalues and have a complete orthogonal set of eigenvectors [2]. Our goal is to study the eigenstructure of F matrices by finding the eigenvalues and their multiplicity, invariant subspaces and projectors to the invariant subspaces.

EIGENVALUES, EIGENSPACES AND PROJECTORS TO EIGENSPACES

The eigenvalues of a matrix are, by definition, the roots of its characteristic polynomial. Here we do not calculate the characteristic polynomial explicitly but relate the powers of **F** to the characteristic polynomial. Let **J** denote the second power of the matrix F, that is $\mathbf{J} = \mathbf{F}^2$. The entries of matrix **J** can be calculated as follows:

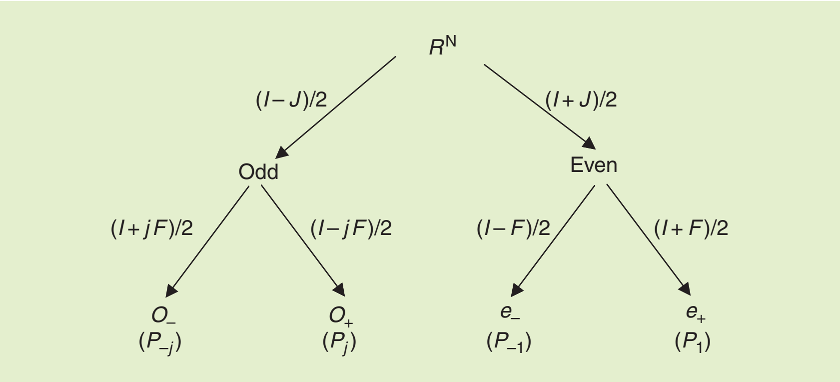
$$[\mathbf{J}]_{k,n} = \sum_{d=0}^{N-1} [\mathbf{F}]_{k,d}[\mathbf{F}]_{d,n}$$
$$= \frac{1}{N}\sum_{d=0}^{N-1} e^{-j\frac{2\pi}{N}(n+k)d} = \delta[(n+k)_N].$$

The notation of $(\cdot)_N$ indicates the modulo N reduction of $(\cdot)$, that is $(n+k)_N \equiv (n+k) \bmod N$. It can be seen that the **J** matrix is a permutation matrix that maps $x[n] \to x[(-n)_N]$. The **J** matrix is called a coordinate inversion or reflection matrix in the literature.

Two coordinate inversion operations executed in a row can be denoted by $\mathbf{J}^2$ or $\mathbf{F}^4$. Since two coordinate inversions result in the identity mapping, $\mathbf{F}^4$ is equal to **I**. If $\mathbf{e}_k$ is an eigenvector of the **F** matrix with the eigenvalue λ_k, then the vector $\mathbf{F}^4\mathbf{e}_k$ should be equal to $\mathbf{F}^4\mathbf{e}_k = \lambda_k^4\mathbf{e}_k$, by the eigenvector definition. Using the identity $\mathbf{F}^4 = \mathbf{I}$ along with the last relation results in the conclusion that the possible values of λ must satisfy $\lambda_k^4 = 1$. Hence the list of possible eigenvalues for the DFT matrix is $\lambda_k = \{1, -1, j, -j\}$.

Having established the list of possible eigenvalues, we construct a $p_1(\lambda)$ polynomial having roots at $\{-1, j, -j\}$ and taking the value of 1 at $\lambda = 1$. Hence, this polynomial takes the value of zero for all except one of the eigenvalues of DFT matrix. This polynomial can be explicitly written as follows

$$p_1(\lambda) = \frac{1}{4}(\lambda^2+1)(\lambda+1)$$
$$= \frac{1}{4}(\lambda^3+\lambda^2+\lambda+1).$$



[FIG1] Decomposition of *N* dimension space into even–odd sub-spaces.

When $\mathbf{F}$ is substituted for λ in $p_1(\lambda)$, we get the $\mathbf{P}_1$ matrix:

$$\mathbf{P}_1 = \frac{1}{4}(\mathbf{F}^3 + \mathbf{F}^2 + \mathbf{F} + \mathbf{I}).$$

When the $\mathbf{P}_1$ matrix is multiplied from right with an eigenvector of the DFT matrix having eigenvalue λ_k, the resultant vector is as given:

$$\mathbf{P}_1\mathbf{e}_k = \begin{cases} \mathbf{0} & \lambda_k = \{-1, j, -j\} \\ \mathbf{e}_k & \lambda_k = 1 \end{cases}.$$

The last relation shows that the eigenvectors of DFT with the eigenvalue of 1 pass through $\mathbf{P}_1$ without any change (mapped to itself) and the other eigenvectors are projected to the zero vector, i.e., elements of null space. Since the eigenvectors of DFT are complete, i.e., span N dimensional space, $\mathbf{e}_k$ vectors form a complete set of eigenvectors for the $\mathbf{P}_1$ matrix. From this information, we can deduce that the matrix $\mathbf{P}_1$ has only two eigenvalues that can be either 0 or 1. This leads to the conclusion that $\mathbf{P}_1$ is a projection matrix [2].

The projection matrices satisfy the relation $\mathbf{P}^2 = \mathbf{P}$. Among the projection matrices, the matrices with the property $\mathbf{P}^T = \mathbf{P}$ are called the orthogonal projectors. With these facts, we can confirm that the matrix $\mathbf{P}_1$ is an orthogonal projector to the range space of DFT eigenvectors having the eigenvalue of 1.

Following the same route, we can write the projectors to four eigenspaces as follows:

$$\mathbf{P}_1 = \frac{1}{4}(\mathbf{F}^3 + \mathbf{F}^2 + \mathbf{F} + \mathbf{I})$$

$$\mathbf{P}_{-1} = \frac{1}{4}(-\mathbf{F}^3 + \mathbf{F}^2 - \mathbf{F} + \mathbf{I})$$

$$\mathbf{P}_j = \frac{1}{4}(j\mathbf{F}^3 - \mathbf{F}^2 - j\mathbf{F} + \mathbf{I})$$

$$\mathbf{P}_{-j} = \frac{1}{4}(-j\mathbf{F}^3 - \mathbf{F}^2 + j\mathbf{F} + \mathbf{I}).$$

Below we present a summary of our current findings along with some new, but easy to establish, results on $\mathbf{P}_k$ matrices:

- $\mathbf{P}_k$ matrices are orthogonal projectors, i.e., $\mathbf{P}_k^2 = \mathbf{P}_k$ and $\mathbf{P}_k^T = \mathbf{P}_k$.
- The projection matrices are complementary ($\mathbf{P}_k\mathbf{P}_l = \mathbf{0}$, $k \neq l$).
- The direct sum of the projection subspaces is $\mathcal{R}^N$.
- The projection subspaces are invariant under DFT, that is, $\mathbf{F}\mathbf{P}_k = \mathbf{P}_k\mathbf{F} = \lambda_k\mathbf{P}_k$.
- $\mathbf{P}_1 + \mathbf{P}_{-1}$ is the projector to the space spanned by even vectors, that is, $\mathbf{E} = \mathbf{P}_1 + \mathbf{P}_{-1} = 1/2(\mathbf{I} + \mathbf{J})$ and $\mathbf{E}\{x[n]\} = 1/2(x[n] + x[(-n)_N])$.
- $\mathbf{P}_j + \mathbf{P}_{-j}$ is the projector to the space spanned by odd vectors, that is, $\mathbf{O} = \mathbf{P}_j + \mathbf{P}_{-j} = 1/2(\mathbf{I} - \mathbf{J})$ and $\mathbf{O}\{x[n]\} = 1/2(x[n] - x[(-n)_N])$.

The results given above can be verified by algebraic multiplication and addition of $\mathbf{P}_k$ matrices. However, we would like to encourage readers not to interpret these results algebraically, but through the concepts of linear spaces, e.g., subspace, range space, and null space. As an example, $\mathbf{P}_1$ is the projector to the space spanned by the eigenvectors with the eigenvalue of one, that is,

$$\mathbf{P}_1 = \sum_{k=1}^{m_1} \mathbf{e}_k^1(\mathbf{e}_k^1)^T. \quad (1)$$

Here m_1 is the multiplicity of the eigenvalue and $\mathbf{e}_k^1$ is the kth eigenvector with the eigenvalue of one. The first and second results given above immediately follow from the definition in (1) and the orthogonality of the eigenvectors with different eigenvalues. The third result is due to the completeness of the eigenvectors. The other results can be interpreted similarly with a little bit of effort.

Up to this point we have studied how to construct the projection matrices for the invariant subspaces of the DFT matrix. It is well known that DFT maps even sequences to even sequences and odd sequences to odd sequences. Hence the subspace of even sequences and odd sequences are invariant under DFT. Here we generalize the invariance property of even and odd subspaces. We show that a vector in $\mathbf{P}_k$ space is mapped to another vector in $\mathbf{P}_k$ space. With this interpretation we can say that $\mathbf{P}_k$ matrices partition even and odd subspaces into two, as shown in Figure 1.

THE MULTIPLICITY OF EIGENVALUES

The eigenvalue multiplicity problem of DFT matrices is known to be a difficult problem. We present a solution to the eigenvalue multiplicity problem using an equally difficult result known as the Gaussian sum. The Gaussian sum identity is given below:

Guassian sum:

$$\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}n^2} = \begin{cases} 1+j & N=4m \\ 1 & N=4m+1 \\ 0 & N=4m+2 \\ j & N=4m+3 \end{cases}.$$

The proof of this result took Gauss two years [3]. Since the original proof of Gauss, it is an ongoing challenge among mathematicians to present new, possibly better, proofs of this result. Interested readers can find four different proofs of Mertens, Kronecker, Schur, and Gauss in [3]. In 1972, J. McClellan solved the eigenvalue multiplicity problem using elementary means [4]. McClellan's solution can be considered as another proof of the Gaussian sum and resides at the intersection of pure and applied

[TABLE I] EIGENVALUE MULTIPLICITY OF $N \times N$ DFT MATRIX.

N	λ = 1	λ = −1	λ = J	λ = −J
4M	M+1	M	M−1	M
4M+1	M+1	M	M	M
4M+2	M+1	M+1	M	M
4M+3	M+1	M+1	M	M+1

mathematics as noted in [5]. McClellan is also known for an optimal filter design technique (Parks-McClellan algorithm) and a multidimensional filter design technique through mapping (McClellan transform) to the DSP community. Here we do not attempt to prove the Gaussian sum and but use the relation for the solution of the DFT eigenvalue multiplicity problem.

It can be noted that the trace of the matrix $\mathbf{P}_k$ is equal to the multiplicity of the eigenvalue with value λ_k. This can be justified from equation (1) by using the identity $\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{BA})$. Another justification can be given by noting that the projection matrices have eigenvalues either zero or one. Therefore the trace, which is the sum of the eigenvalues, is equal to the number of eigenvalues with value one.

The trace of the projection matrices can be written as follows:

$$\text{trace}\{\mathbf{P}_1\} = \frac{1}{4}\left\{\frac{2}{\sqrt{N}}\sum_{n=0}^{N-1}\cos\left(\frac{2\pi}{N}n^2\right) + \text{trace}\{\mathbf{J}\} + N\right\}$$

$$\text{trace}\{\mathbf{P}_{-1}\} = \frac{1}{4}\left\{-\frac{2}{\sqrt{N}}\sum_{n=0}^{N-1}\cos\left(\frac{2\pi}{N}n^2\right) + \text{trace}\{\mathbf{J}\} + N\right\}$$

$$\text{trace}\{\mathbf{P}_j\} = \frac{1}{4}\left\{\frac{2}{\sqrt{N}}\sum_{n=0}^{N-1}\sin\left(\frac{2\pi}{N}n^2\right) - \text{trace}\{\mathbf{J}\} + N\right\}$$

$$\text{trace}\{\mathbf{P}_{-j}\} = \frac{1}{4}\left\{\frac{2}{\sqrt{N}}\sum_{n=0}^{N-1}\sin\left(\frac{2\pi}{N}n^2\right) - \text{trace}\{\mathbf{J}\} + N\right\}.$$

Using the Gaussian summation, the trace of each matrix can be easily calculated and the eigenvalue multiplicity of DFT matrices can be found as shown in Table 1 (we note that the trace of the $N \times N$ $\mathbf{J}$ matrix is equal to one and two for odd and even values of N, respectively).

EIGENVECTORS OF DFT MATRIX

The eigenvector set of DFT matrices for $N \geq 4$ are not unique due to the eigenvalue multiplicity problem as shown in Table 1. The table indicates that there are infinitely many eigenvector sets of DFT matrix.

An eigenvector set of DFT can be easily constructed using projection matrices. Since the projection spaces are invariant under DFT operation, that is, $\mathbf{FP}_k = \lambda_k \mathbf{P}_k$, the columns of projection matrix $\mathbf{P}_k$ are the eigenvectors of DFT. Unfortunately, this eigenvector set does have the orthogonality property. If the orthogonality of eigenvectors is desired, one can apply the Gram-Schmidt procedure over the columns of $\mathbf{P}_k$. This operation can be done with a few lines of MATLAB code as shown below:

WE HAVE EXAMINED THE STRUCTURE OF DFT EIGENSPACES AND USED THE PROJECTORS TO THE INVARIANT SPACES TO ESTABLISH SOME CONNECTIONS WITH THE RELATIVES OF THE FOURIER TRANSFORM.

```
>> N=7; F = 1/
   sqrt(N)*dftmtx(N);
>> P1 = 0.25 * (F^3 + F^2 +
   F + eye(N));
>> E1 = orth(P1);
```

To get a distinct set of orthogonal eigenvectors with eigenvalue of 1, we can modify the last line as follows:

```
>> E1=orth(P1*randn(N,N))
```

An alternative approach is to define a commuting matrix $\mathbf{K}$ through an arbitrary but a full-rank matrix $\mathbf{M}$ as shown below:

$$\mathbf{K} = \mathbf{M} + \mathbf{FMF}^{-1} + \mathbf{F}^2\mathbf{MF}^{-2} + \mathbf{F}^3\mathbf{MF}^{-3}.$$

It is easy to show that matrices $\mathbf{K}$ and $\mathbf{F}$ commute; that is, $\mathbf{FK} = \mathbf{KF}$, therefore $\mathbf{K}$ and $\mathbf{F}$ have a common eigenvector set, [7, p. 52]. In other words, by finding the eigenvectors of $\mathbf{K}$ matrix, we can also get the eigenvectors of the DFT matrix. This technique has been applied to derive the eigenvectors of the DFT with some desirable features. In [8] and [9], the discrete equivalents of Hermite-Gaussian functions (which are the eigenfunctions of continuous Fourier transform) are defined by a proper choice of $\mathbf{M}$ matrix.

EXTENSIONS

Up to this point we have presented results on a one-dimensional conventional DFT matrix. In this section, we extend the earlier results to non-conventional DFT matrices and multidimensional DFT matrices, and establish some connections with other relatives of the Fourier transform.

EIGENSTRUCTURE OF OFFSET DFT

The offset DFT is a generalization of the conventional DFT. Its definition is given as follows:

$$[\mathbf{F}_{a,b}]_{k,n} = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi}{N}(k-a)(n-b)}.$$

The offset DFT has two parameters (a and b) that can be freely selected. It can be shown that the offset DFT matrix is unitary and reduces to the conventional DFT when $a = b = 0$ [10]. The special case of $a = b = 1/2$, is called an odd-time odd-frequency DFT and was studied in [11]. The eigenstructure of the offset DFT has been shown to be closely related to ordinary DFT for the special case of $a = b = 1/2$, [12]. The other cases are a little more complicated and studied under the categories of $a + b =$ integer and $a + b \neq$ integer. Further details can be found in [13].

EIGENSTRUCTURE OF MULTIDIMENSIONAL DFT

By definition, the multidimensional DFT is a separable transformation. Hence a two-dimensional DFT operation can be interpreted as the cascade application of a one-dimensional DFT to the columns of the input (a matrix) followed by the application of DFT to the rows of the resultant matrix. The separability property aids in identifying the eigenstructure of multidimensional DFT.

It can be noted that the following $M \times N$ rank-1 matrix is an eigenmatrix of two dimensional DFT with the eigenvalue $\lambda_x \lambda_y$:

$$\mathbf{E} = \mathrm{e}_x \mathrm{e}_y^T. \quad (2)$$

Here e_x is an eigenvector of $M \times M$ one-dimensional DFT matrix with the eigenvalue λ_x and e_y is an eigenvector of $N \times N$ one-dimensional DFT matrix with the eigenvalue λ_y. From this discussion, it can be noted that the set of eigenvalues of a two-dimensional DFT is identical to the corresponding set of a one-dimensional transform. The results on a two-dimensional DFT can be easily extended to multidimensions. More details can be found in [14].

RELATIONS TO OTHER TRANSFORMS

The projectors to the invariant spaces of the DFT can be useful to characterize other relatives of the Fourier transform. The following lines show the relation between the projectors and DFT, Hartley transform, identity, and coordinate inversion operations respectively:

$$\mathbf{F} = \mathbf{P}_1 - \mathbf{P}_{-1} + j\mathbf{P}_j - j\mathbf{P}_{-j}$$

$$\mathbf{H} = \mathbf{P}_1 - \mathbf{P}_{-1} - \mathbf{P}_j + \mathbf{P}_{-j}$$

$$\mathbf{I} = \mathbf{P}_1 + \mathbf{P}_{-1} + \mathbf{P}_j + \mathbf{P}_{-j}$$

$$\mathbf{J} = \mathbf{P}_1 + \mathbf{P}_{-1} - \mathbf{P}_j - \mathbf{P}_{-j}.$$

It can be noted that the projectors define an algebra for the relatives of the Fourier transform. As an illustrative example, the transformation formed by the cascade application of a Hartley transform and a DFT transform, that is, an **FH** matrix can be expressed in terms of projectors as follows:

THE PROJECTORS TO THE INVARIANT SPACES OF THE DFT CAN BE USEFUL TO CHARACTERIZE OTHER RELATIVES OF THE FOURIER TRANSFORM.

$$\begin{aligned} \mathbf{FH} &= (\mathbf{P}_1 - \mathbf{P}_{-1} + j\mathbf{P}_j - j\mathbf{P}_{-j}) \\ &\quad \times (\mathbf{P}_1 - \mathbf{P}_{-1} - \mathbf{P}_j + \mathbf{P}_{-j}) \\ &= \mathbf{P}_1 + \mathbf{P}_{-1} - j\mathbf{P}_j - j\mathbf{P}_{-j} \\ &= \frac{1}{2}(\mathbf{I} + \mathbf{J}) - \frac{j}{2}(\mathbf{I} - \mathbf{J}) \\ &= \mathbf{E} - j\mathbf{O}. \end{aligned}$$

From this result, we can conclude that the cascade operation of Hartley and DFT is equivalent to expressing even and odd parts of the input and combining them together as the real and imaginary parts of the output.

The fractional powers or any other function of **F** can also be defined through the projectors. We illustrate the idea on the square root of a DFT matrix. The square root or one half power of a DFT matrix can be defined as follows $\mathbf{F}^{\frac{1}{2}} \triangleq \sqrt{1}\mathbf{P}_1 + \sqrt{-1}\mathbf{P}_{-1} + \sqrt{j}\mathbf{P}_j + \sqrt{-j}\mathbf{P}_{-j}$. Since the square root operation is one-to-many, that is $\sqrt{1} = \{1, -1\}$, the proposed definition is not unique unless a branch-cut for every square root is specified. A possible definition is $\mathbf{F}^{\frac{1}{2}} \triangleq \mathbf{P}_1 + j\mathbf{P}_{-1} + \frac{1+j}{2}\mathbf{P}_j + \frac{1-j}{2}\mathbf{P}_{-j}$. One can easily note that $\mathbf{F}^{\frac{1}{2}}\mathbf{F}^{\frac{1}{2}} = \mathbf{F}$ as expected. More information on the fractional Fourier transform and the details of the definition multiplicity problem can be found in [15].

CONCLUSIONS

We have examined the structure of DFT eigenspaces and used the projectors to the invariant spaces to establish some connections with the relatives of the Fourier transform. The presented results are heavily based on the properties of projectors that can also be of interest on their own due to their strong algebraic structure and important geometric interpretations.

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