Problem:

Problem set-up: $x[n]$ is a given *causal* sequence. $H(z) = \frac{Bq(z)}{A_p(z)}$ is the impulse response of an LTI system with q zeros and p poles where

$$B_q(z) = b_0 + b_1 z^{-1} + \ldots + b_q z^{-q}$$
$$A_p(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_p z^{-p}$$

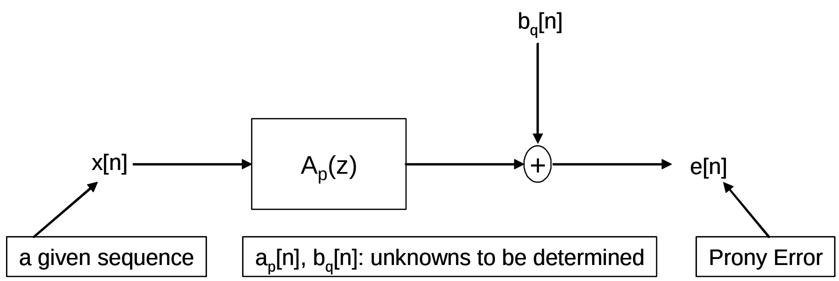
The goal is to set $A_p(z)$ and $B_q(z)$ so that $h[n]$ approximates $x[n]$ in some sense.

Prony's Method (2nd Derivation) (Hayes p.144)

Let $e'[n] = x[n] - h[n]$ then $E'(z) = X(z) - H(z) = X(z) - \frac{Bq(z)}{A_p(z)}$. We call $E(z) = E'(z)A_p(z)$ as the Prony error:

$$E(z) = X(z)A_p(z) - B_q(z)$$

The following shows the system producing the Prony error.



The Prony error can be expressed as follows:

$$e[n] = a_p[n] * x[n] - b_q[n] = \begin{cases} x[n] + \sum_{l=1}^{P} a_p[l]x[n-l] - b_q[n] & 0 \le n \le q \\ x[n] + \sum_{l=1}^{P} a_p[l]x[n-l] & n \ge q+1 \end{cases}$$

For any given set of $a_p[n]$ coefficients, it is clear that one can set $b_q[n]$ such that $e[n] = 0$ for $0 \le n \le q$. Hence Prony cost function optimizes over $a_p[n]$ first and it is defined as

$$J(a_p) = \sum_{n=q+1}^{\infty} |e[n]|^2 \text{ where } q \text{ is the number of poles}$$

Taking the partial derivative with respect to $a_p^*[k]$, we get

$$
\begin{aligned}
\frac{\partial}{\partial a_p^*[k]} J(a_p) &= \sum_{n=q+1}^{\infty} e[n] \frac{\partial}{\partial a_p^*[k]} e^*[n] = \sum_{n=q+1}^{\infty} e[n] x^*[n-k] \\
&= \sum_{n=q+1}^{\infty} \left(x[n] + \sum_{l=1}^{P} a_p[l] x[n-l] \right) x^*[n-k] \\
&= \sum_{n=q+1}^{\infty} x[n] x^*[n-k] + \sum_{l=1}^{P} a_p[l] \sum_{n=q+1}^{\infty} x[n-l] x^*[n-k] \\
&= r_x(k,0) + \sum_{l=1}^{P} a_p[l] r_x(k,l)
\end{aligned}
$$

Here we define the auto-correlation function

$$
r_x(k,l) = \sum_{n=q+1}^{\infty} x[n-l] x^*[n-k]
$$

Note that $r_x(k,l)$ can not be written as a function of $k-l$. (Check whether $r_x(0,0) \stackrel{?}{=} r_x(1,1)$)

Then equating $\frac{\partial}{\partial a_p^*[k]} J(a_p) = 0$ for $1 \le k \le p$, we get the following system of equations:

$$
\begin{bmatrix}
r_x(1,1) & r_x(1,2) & \dots & r_x(1,P) \\
r_x(2,1) & r_x(2,2) & \dots & r_x(2,P) \\
\vdots & \vdots & & \vdots \\
r_x(P,1) & r_x(P,2) & \dots & r_x(P,P)
\end{bmatrix}
\begin{bmatrix}
a_p[1] \\ a_p[2] \\ \vdots \\ a_p[P]
\end{bmatrix}
= -
\begin{bmatrix}
r_x(1,0) \\ r_x(2,0) \\ \vdots \\ r_x(P,0)
\end{bmatrix}
$$

From the equation system, we can solve for the unknown $a_p[k]$'s.

Previously, we have solved the same problem via the Least Squares solution of an overdetermined equation system. Let's compare that solution with the one involving $r_x(k,l)$'s:

$$
\begin{bmatrix}
x[0] & 0 & 0 & \dots & 0 \\
x[1] & x[0] & 0 & \dots & 0 \\
x[2] & x[1] & x[0] & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
x[q] & x[q-1] & x[q-2] & \dots & x[q-p-1] \\
x[q+1] & x[q] & x[q-1] & \dots & x[q-p] \\
\vdots & \vdots & \vdots & & \vdots \\
x[N] & x[N-1] & x[N-2] & \dots & x[N-p-1]
\end{bmatrix}
\begin{bmatrix}
1 \\ a_p[1] \\ \vdots \\ a_p[P]
\end{bmatrix}
= -
\begin{bmatrix}
b_q[0] \\ b_q[1] \\ b_q[2] \\ \vdots \\ b_q[q] \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

Remember, we use the bottom part of the matrix for the solution of $a_p[k]$'s, that is

$$\begin{bmatrix} x[q+1] & x[q] & x[q-1] & \dots & x[q-p] \\ x[q+2] & x[q+1] & x[q] & \dots & x[q-p+1] \\ \vdots & \vdots & \vdots & & \vdots \\ x[N] & x[N-1] & x[N-2] & \dots & x[N-p-1] \end{bmatrix} \begin{bmatrix} 1 \\ a_p[1] \\ \vdots \\ a_p[P] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Let’s leave the unknowns on the left hand side of the equation system:

$$\begin{bmatrix} x[q] & x[q-1] & \dots & x[q-p] \\ x[q+1] & x[q] & \dots & x[q-p+1] \\ \vdots & \vdots & & \vdots \\ x[N-1] & x[N-2] & \dots & x[N-p-1] \end{bmatrix} \begin{bmatrix} a_p[1] \\ a_p[2] \\ \vdots \\ a_p[P] \end{bmatrix} = -\begin{bmatrix} x[q+1] \\ x[q+2] \\ \vdots \\ x[N] \end{bmatrix}$$

The equation system is in the standard form of $Ax = b$. The LS solution is $x^{LS} = (A^H A)^{-1} A^H b$ or is the solution of the following equation system:

$$(A^H A) x^{LS} = A^H b$$

Here

$$A = \begin{bmatrix} x[q] & x[q-1] & \dots & x[q-p] \\ x[q+1] & x[q] & \dots & x[q-p+1] \\ \vdots & \vdots & & \vdots \\ x[N-1] & x[N-2] & \dots & x[N-p-1] \end{bmatrix}, \quad b = \begin{bmatrix} x[q+1] \\ x[q+2] \\ \vdots \\ x[N] \end{bmatrix}$$

It is possible to check that the k’th row, the l’th column entry of $A^H A$ is $r_x(k, l)$. (Note that the k’th row and l’th column entry of $A^H A$ is the inner product of the k’th column and l’th column of A.)