MATH 782, Descriptive Set Theory, Homework 4

1. (3+5+5+6+5 pts) In this homework, we shall be interested in two-player perfect information games of length ω played on $2 = \{0, 1\}$. Recall that, in Homework 1, we have constructed the Polish space of $\mathcal{G}$ of countable graphs. Let $S \subseteq \mathcal{G}$ be a set and let $f : \mathbb{N} \to [\mathbb{N}]^2$ be a fixed enumeration of two element subsets of $\mathbb{N}$.

Alice and Bob are playing the graph construction game GC_S with payoff set S which is played as follows:

- Alice starts the game.
- Alice and Bob move alternately.
- At their turn, each player chooses an element of $2 = \{0, 1\}$.
- The game ends after ω moves.

Suppose that, in a play of the game, players make the following moves.

Alice	e_0		e_2		e_4		$\dots$
Bob		e_1		e_3		e_5	$\dots$

Consider the countable graph $E \in \mathcal{G}$,

$$E = \bigcup_{\substack{n\in\mathbb{N} \\ e_n = 1}} \{(k,\ell),(\ell,k) \in \mathbb{N}\times\mathbb{N} : \ f(n) = \{k,\ell\}\}$$

In other words, at the n-th stage of the game GC_S, the corresponding player decides whether or not to include the n-th edge $f(n)$ in the resulting graph $(\mathbb{N}, E)$, by choosing 1 to include and choosing 0 to not include. The winning condition is that Alice wins the game GC_S if and only if $E \in S$.

a) Let $\mathbb{E}$ denote the isomorphism relation on $\mathcal{G} \times \mathcal{G}$. Find $F \in \mathcal{G}$ such that Bob has a winning strategy in $\mathrm{GC}_{[F]_{\mathbb{E}}}$.[1]

b) Show that there exists $S \subseteq \mathcal{G}$ for which the game GC_S is not determined.

c) Let G be a finite graph. Show that the game $\mathrm{GC}_{\mathcal{G}_{G\text{-free}}}$ is determined where

$$\mathcal{G}_{G\text{-free}} = \{E \in \mathcal{G} : \ (\mathbb{N}, E) \text{ does not contain an isomorphic copy of } G \text{ as a subgraph}\}$$

Recall that we have learned the basic theory of two-player perfect information games of length ω played on $\mathbb{N}$ in class. For each $S \subseteq \mathbb{N}^{\mathbb{N}}$, let G_S denote the corresponding classical game.

d) Show that the graph construction games are equivalent to the classical games in the following sense: There exists a continuous function $f : \mathbb{N}^{\mathbb{N}} \to \mathcal{G}$ such that, for all $S \subseteq \mathcal{G}$, the classical game $G_{f^{-1}(S)}$ is determined if and only if the game GC_S is determined.

e) Show that $\mathrm{GC}_{[E]_{\mathbb{E}}}$ is determined for every $E \in \mathcal{G}$, provided that there exists a Ramsey cardinal.

---

[1] It is an exercise -actually, a part of the question- to find the suitable definitions of a strategy, a winning strategy, a play, a determined game etc. for the graph construction game.