MATH 782, Descriptive Set Theory, Homework 3

1. (4+5+4+5+6 pts) Recall that, in Homework 1, we have constructed the Polish space of $\mathcal{G}$ of countable graphs. In what follows, all topological spaces are endowed with their Borel σ-algebra, all product spaces are endowed with their product σ-algebra and all subsets of these measurable spaces are endowed with the restrictions of the relevant σ-algebras.

a) Prove that the degree map $d : \mathcal{G} \to \mathbb{N} \cup \{\infty\}$ given by

$$d(E) = \sup_{n\in\mathbb{N}} \deg_E(n)$$

is a $\mathbf{\Sigma^0_3}$-map, that is, the inverse images of open sets are in $\mathbf{\Sigma^0_3}(\mathcal{G})$.

b) Prove that the set

$$A = \{E \in \mathcal{G} :\ (\mathbb{N}, E) \text{ contains an isomorphic copy of each finite graph}\}$$

is in $\mathbf{\Pi^0_2}(\mathcal{G}) \setminus \mathbf{\Sigma^0_2}(\mathcal{G})$.

c) Prove that the left action[1] $S_\infty \curvearrowright \mathcal{G}$ of the infinite symmetric group given by

$$(\varphi \cdot E)(i, j) = 1 \text{ iff } E(\varphi^{-1}(i), \varphi^{-1}(j)) = 1 \text{ for all } i, j \in \mathbb{N}$$

is continuous as a map from $S_\infty \times \mathcal{G}$ to $\mathcal{G}$.

d) Prove the isomorphism relation[2]

$$\mathbb{E} = \{(E, F) \in \mathcal{G} \times \mathcal{G} :\ (\mathbb{N}, E) \cong (\mathbb{N}, F)\}$$

is a $\mathbf{\Sigma^1_1}$-subset of $\mathcal{G} \times \mathcal{G}$.

e) Prove that the set

$$A = \{E \in \mathcal{G} :\ (\mathbb{N}, E) \text{ contains an isomorphic copy of the complete graph } K_\mathbb{N} \text{ as a subgraph}\}$$

is a $\mathbf{\Sigma^1_1}$-complete subset of $\mathcal{G}$.

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[1] This is called the *logic action* of S_∞ since its relations to the model theory of countable structures in countable relational languages in the infinitary logic $\mathcal{L}_{\omega_1\omega}$. You can see Gao's "Invariant Descriptive Set Theory" for a detailed account.

[2] It turns out that the relation is "as complex as possible" in a very precise sense and also is a $\mathbf{\Sigma^1_1}$-complete subset of $\mathcal{G} \times \mathcal{G}$. However, we shall not attempt to formalize and prove these non-trivial facts in this homework.