MATH 782, Descriptive Set Theory, Homework 2

1. (5+5+4+6+4 pts) In Homework 1, we have constructed the Polish space of $\mathcal{G}$ of countable graphs. In what follows, all topological spaces are endowed with their Borel σ-algebra and all subsets of these measurable spaces are endowed with the restrictions of the relevant σ-algebras.

a) Determine whether or not the degree map $d : \mathcal{G} \to \mathbb{N} \cup \{\infty\}$ given by

$$d(E) = \sup_{n \in \mathbb{N}} \deg_E(n)$$

is a Borel map.

b) Prove that the set

$$\mathcal{CG} = \{E \in \mathcal{G} : \ (\mathbb{N}, E) \text{ is a connected graph}\}$$

is a Borel subset of $\mathcal{G}$.

A motto that you often hear from set theorists is that **EXPLICIT=BOREL**. More specifically, whenever you have a construction that you informally think is explicit, that construction turns out to be Borel almost all the time once the setting is appropriately translated into the realm of Polish space. In this homework, we shall see an example of this phenomenon.

Recall that a connected graph is called a *tree* if it has no cycles. Given a connected graph $\mathcal{G}$, a *spanning tree* of $\mathcal{G}$ is a subgraph that is a tree and that contains all vertices of $\mathcal{G}$. By the axiom of choice, we know that every connected graph has a spanning tree. Indeed, the existence of a spanning tree for arbitrary infinite connected graphs is equivalent to the axiom of choice. That said, the use of the axiom of choice may not be necessary in certain cases, which actually includes our case in this homework.

There are various algorithms to find a spanning tree. In the following questions, we shall gradually show that one of these algorithms defines a Borel map on $\mathcal{CG}$ as our motto foresees. Before we describe our algorithm, fix an enumeration of $\mathbb{N} \times \mathbb{N} = \{(a_n, b_n)\}_{n \in \mathbb{N}}$ of the set of all possible directed edges. Recall that, given vertices $a, b \in \mathbb{N}$, we think of $(a, b) \in \mathbb{N} \times \mathbb{N}$ as the directed edge from a to b. Therefore, if 779 and 782 are adjacent in the graph $(\mathbb{N}, E)$, then we have both $(779, 782), (782, 779) \in E$. Indeed, this is why we required E to be a symmetric relation in the construction of $\mathcal{G}$.

Here a modified version of Kruskal's algorithm to find a spanning tree of a graph of the form $(\mathbb{N}, E)$: Given a graph $(\mathbb{N}, E)$, we shall construct a subgraph $(\mathbb{N}, F)$ of $(\mathbb{N}, E)$ via recursion on $n \in \mathbb{N}$ by determining whether or not to include (a_n, b_n) in this subgraph as follows:

- Start with the subgraph $(\mathbb{N}, \emptyset)$, that is, the subgraph that has all the vertices but has no edges.
- At Stage $n \in \mathbb{N}$, check whether or not $(a_n, b_n) \in E$.
- If this *is not* the case, skip to Stage $n + 1$ without changing the subgraph constructed up to this stage.

- If this *is* the case, then we must also have $(b_n, a_n) \in E$ and now, check whether adding the directed edges $(a_n, b_n), (b_n, a_n)$ creates a cycle in the subgraph constructed up to this stage:
  - If adding these directed edges *does* create a cycle, skip to Stage $n+1$ without changing the subgraph constructed up to this stage.
  - If adding these directed edges *does not* create a cycle, then update the subgraph constructed up to this stage by adding these directed edges and move on to Stage $n+1$.[1]

A moment's thought[2] shall reveal that, after ω stages, the subgraph $(\mathbb{N}, F)$ constructed by this procedure will be a spanning tree of $(\mathbb{N}, E)$. We shall prove that this procedure defines a Borel map on $\mathcal{CG}$.

c) Show that the set $\mathrm{Cyc}(\{320, 406, 779, 782\})$ defined as

$$\{E \in \mathcal{CG} : \ (\mathbb{N}, E) \text{ has a cycle of length 4 passing through } 320,\ 406,\ 779,\ 782\}$$

is a Borel subset of $\mathcal{CG}$.

d) Let $(i, j) \in \mathbb{N} \times \mathbb{N}$. Show that

$$B = \{E \in \mathcal{CG} : \text{Kruskal's algorithm applied to } (\mathbb{N}, E) \text{ results in } (\mathbb{N}, F) \text{ and } F(i, j) = 0\}$$

is a Borel subset of $\mathcal{CG}$.

e) Prove that the map $\varphi : \mathcal{CG} \to \mathcal{CG}$ with $\varphi(E) = F$, where $(\mathbb{N}, F)$ is the spanning tree obtained from $(\mathbb{N}, E)$ via applying Kruksal's algorithm above, is a Borel map.

Moral of the story. Any explicit procedure on countable, or more generally, "separable" objects that uses a *countable amount of information* will most likely define a Borel map once you appropriately code it on a Polish space.

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[1] Observe that, since (b_n, a_n) is equal to (a_k, b_k) for some $k \in \mathbb{N}$, we are actually checking twice whether or not adding these directed edges create a cycle during this procedure, at Stage n and Stage k. However, this will create no problem because if these edges have been added before, then "readding" them simply does nothing and if these edged have not been added before, then they cannot be added later at a stage.

[2] Here is how you can observe that this procedure will find a spanning tree: You can prove by induction that the subgraph constructed any stage has no cycle. Hence the resulting graph after ω stages will contain no cycle because a cycle that appears after ω stages would have appeared at a finite stage due to a cycle being a finite sequence of edges. Moreover, the resulting graph would be connected. Suppose not, say, $k, \ell \in \mathbb{N}$ are vertices in two different non-empty connected components of the resulting graph. Since $(\mathbb{N}, E)$ is connected, we know that there exists a path $P \in (\mathbb{N} \times \mathbb{N})^{<\mathbb{N}}$ in $(\mathbb{N}, E)$ in connecting k to ℓ. This path can be decomposed into two subpaths $P = P_1 \frown P_2$ where the path P_1 goes through vertices that are in the connected component of k and the first edge of P_2 is of the form (a, b) where a is in the connected component of k but b is not in the connected component of k. But by construction, since the edge (a, b) has been discarded, it must have been creating a cycle during the procedure. However, this cannot happen since the connected component of a, which is equal to the connected component of k, and the connected component of b are disjoint and have no cycles.