MATH 782, Descriptive Set Theory, Homework 1

1. (6+6+6+3+3 pts) In this question, we shall construct the Polish space $\mathcal{G}$ of countable graphs and make some basic observations.

Recall that a graph is a pair (V, E) where V is the vertex set and $E \subseteq V \times V$ is an irreflexive symmetric relation on V that is called the edge relation. The class of countable graphs is indeed a proper class. However, for the purpose of coding these graphs as members of a set, we may assume without loss of generality that the underlying vertex set V is equal to $\mathbb{N}$.

From now on, we shall only be interested in countable graphs of the form $(\mathbb{N}, E)$ where $E \subseteq \mathbb{N} \times \mathbb{N}$ is an irreflexive symmetric relation. Observe that such a countable graph is completely determined by its edge relation E. In what follows, you must identify relations on $\mathbb{N}$ as elements of the Cantor space $2^{\mathbb{N}\times\mathbb{N}}$ using the canonical bijection $S \mapsto \chi_S$ between $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ and $2^{\mathbb{N}\times\mathbb{N}}$. For notational simplicity, in order to avoid the cumbersome notation of writing χ_E repeatedly, you may write $E(i, j) = 1$ and $E(i, j) = 0$ instead of $(i, j) \in E$ and $(i, j) \notin E$ respectively.

a) Prove that the set

$$\mathcal{G} = \{E \in 2^{\mathbb{N}\times\mathbb{N}} : \ E \text{ is the edge relation of a countable graph on } \mathbb{N}\}$$

is a Polish subspace of the Cantor space $2^{\mathbb{N}\times\mathbb{N}}$.

b) Determine whether or not the function $d : \mathcal{G} \to \mathbb{N} \cup \{\infty\}$ given by

$$d(E) = \sup_{n\in\mathbb{N}} \deg_E(n)$$

is continuous, where $\deg_E(n)$ denotes the degree of the vertex n in the graph $(\mathbb{N}, E)$ and the set $\mathbb{N} \cup \{\infty\}$ is endowed with the discrete topology.

c) Let G be a finite graph. Prove that the set

$$\mathcal{G}_{G\text{-free}} = \{E \in \mathcal{G} : \ (\mathbb{N}, E) \text{ does not contain an isomorphic copy of } G \text{ as a subgraph}\}$$

is a nowhere subset of $\mathcal{G}$.

A property $\varphi(x)$ of countable graphs is said to hold *generically* if it holds on a comeager subset of $\mathcal{G}$, that is,

$$\{E \in \mathcal{G} : \varphi(\mathbb{N}, E) \text{ holds}\}$$

is comeager. In this case, we shall say that a generic countable graph has the property $\varphi(x)$.

d) Prove that a generic countable graph contains isomorphic copies of all finite graphs as subgraphs.

e) Using the Baire category theorem, show that there exist countable graphs that contain isomorphic copies of all finite graphs as subgraphs.
Remark. Even though one can explicitly construct such graphs easily, the point of the question is to be able to use the Baire category theorem to deduce this fact.