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FULL NAME	STUDENT ID	DURATION 180+ϵ MINUTES
1 QUESTION 3 PAGES		TOTAL 100 POINTS

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Constructions in mathematics often depend on specific enumerations and choices of the building blocks of the objects to be constructed. Therefore, it is natural question to ask whether or not these constructions can be done *uniformly*. In many cases, this turns out to be not the case. In other words, there is an *inevitable non-uniformity* in mathematics, as my advisor Simon Thomas once put it. Throughout its evolution, descriptive set theory has provided tools that explain this non-uniformity phenomenon.

In this exam, you shall learn one of the simplest results on this theme and prove Friedman's Borel diagonalization theorem. Consider this final exam as a story to answer the following question: **Can you prove Cantor's theorem in a uniform fashion?**

Every mathematician knows Cantor's diagonalization argument. Consider the Cantor space $\mathcal{C} = 2^{\mathbb{N}}$. The procedure in Cantor's diagonalization argument can be seen as the map $f : \mathcal{C}^{\mathbb{N}} \to \mathcal{C}$ given by

$$f\big((\mathbf{a}^n)_{n\in\mathbb{N}}\big) = \big(1 - a^k_k\big)_{k\in\mathbb{N}}$$

$$f\Bigg((a^0_0, a^0_1, a^0_2, \dots), (a^1_0, a^1_1, a^1_2, \dots), (a^2_0, a^2_1, a^2_2, \dots), \dots\Bigg) = (1 - a^0_0, 1 - a^1_1, 1 - a^2_2, \dots)$$

In other words, f takes a sequence $(\mathbf{a}^n)_{n\in\mathbb{N}}$ of binary sequences and produces a binary sequence $f\big((\mathbf{a}^n)_{n\in\mathbb{N}}\big)$ that diagonalizes against $(\mathbf{a}^n)_{n\in\mathbb{N}}$ so that $f\big((\mathbf{a}^n)_{n\in\mathbb{N}}\big) \notin \{\mathbf{a}^n\}_{n\in\mathbb{N}}$. With this point of view, Cantor's diagonalization argument actually becomes an explicit construction procedure rather than a reductio ad absurdum argument. As we expect from any other explicit construction, we expect it to be a Borel construction.

(a) Show that f is a Borel map.

At this point, we realize an important feature, or perhaps, a deficiency of Cantor's argument. The binary sequence $f\big((\mathbf{a}^n)_{n\in\mathbb{N}}\big)$ produced by the diagonal argument is not determined by the *set* $\{\mathbf{a}^n\}_{n\in\mathbb{N}}$ but rather depends on the specific *sequence* $(\mathbf{a}^n)_{n\in\mathbb{N}}$. At this point, arises the following natural question: Can you prove Cantor's theorem in a uniform fashion so that the produced sequence only depends on the relevant set?

More formally, does there exist a map $\varphi : \mathcal{C}^{\mathbb{N}} \to \mathcal{C}$ such that

i. $\varphi\big((\mathbf{a}^n)_{n\in\mathbb{N}}\big) \notin \{\mathbf{a}^n\}_{n\in\mathbb{N}}$, that is, φ "diagonalizes" against the sequence $(\mathbf{a}^n)_{n\in\mathbb{N}}$, and

ii. $\varphi\big((\mathbf{a}^n)_{n\in\mathbb{N}}\big) = \varphi\big((\mathbf{b}^n)_{n\in\mathbb{N}}\big)$ whenever $\{\mathbf{a}^n\}_{n\in\mathbb{N}} = \{\mathbf{b}^n\}_{n\in\mathbb{N}}$, that is, φ produces the same binary sequence whenever it is applied to those sequences that determine the same set of binary sequences,

for all $(\mathbf{a}^n)_{n\in\mathbb{N}}, (\mathbf{b}^n)_{n\in\mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$?

The answer to this question is clearly *yes* if one allows *arbitrary* maps: Consider the equivalence relation on $\mathcal{C}^{\mathbb{N}}$ that relates those sequences that define the same set of binary sequences. Using the axiom of choice, choose one representative from each equivalence class and send the elements of each equivalence class to the sequence that is obtained from applying Cantor's original diagonalization map f to the representative sequence of this equivalence class. However, since we use the axiom of choice, the map φ produced in this fashion may not be well-behaved and could be a pathological object. Can we do this *explicitly* without using the axiom of choice?

Since descriptive set theorists often tend to associate *explicit* with *Borel*, the question above can be rephrased in the following precise form: Does there exist a *Borel* map $\varphi : \mathcal{C}^{\mathbb{N}} \to \mathcal{C}$ that satisfies the properties i. and ii. above?

Friedman's Borel diagonalization theorem tells us that the answer to this question is *no*. In other words, you cannot prove Cantor's theorem in an explicit and uniform fashion. We shall now prove this marvelous fact. Instead of working with the Cantor space $\mathcal{C}$, we shall work with the good old real numbers $\mathbb{R}$. However, the result can easily be transferred to the Cantor space $\mathcal{C}$ via the Borel isomorphism theorem.

Theorem. Let $\varphi : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be a Borel map such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{N}}$,

$$\text{if } \{x_n\}_{n\in\mathbb{N}} = \{y_n\}_{n\in\mathbb{N}}, \text{ then } \varphi\big((x_n)_{n\in\mathbb{N}}\big) = \varphi\big((y_n)_{n\in\mathbb{N}}\big).$$

Then there exists $\mathbf{a} \in \mathbb{R}^{\mathbb{N}}$ such that $\varphi(\mathbf{a}) \in \{a_n\}_{n\in\mathbb{N}}$.

Proof. For each $q \in \mathbb{Q}$, set $A_q = \{\mathbf{x} \in \mathbb{R}^{\mathbb{N}} : \varphi(\mathbf{x}) < q\}$.

In addition to the product space $\mathbb{R}^{\mathbb{N}}$ where $\mathbb{R}$ is endowed with its usual topology, consider the product space $\widehat{\mathbb{R}}^{\mathbb{N}}$ where $\widehat{\mathbb{R}}$ is the set of real numbers endowed with the *discrete* topology. Observe that $\widehat{\mathbb{R}}^{\mathbb{N}}$ is not a Polish space, however, it is a completely metrizable space and hence, the Baire category theorem holds for the space $\widehat{\mathbb{R}}^{\mathbb{N}}$. Our first target is to prove the following lemma.

Lemma. For each $q \in \mathbb{Q}$, either A_q or A_q^c is meager in $\widehat{\mathbb{R}}^{\mathbb{N}}$.
Proof of the lemma. Let $q \in \mathbb{Q}$.

(b) Show that A_q is Borel in the product spaces $\mathbb{R}^{\mathbb{N}}$. Using this fact, deduce that A_q is also Borel in the product space $\widehat{\mathbb{R}}^{\mathbb{N}}$.

Since A_q is Borel in $\widehat{\mathbb{R}}^{\mathbb{N}}$, it has the Baire property and so, there exists an open subset $U \subseteq \widehat{\mathbb{R}}^{\mathbb{N}}$ such that $A_q \Delta U$ is meager in $\widehat{\mathbb{R}}^{\mathbb{N}}$. For each permutation $\sigma \in \mathrm{Fin}(\mathbb{N}) \subseteq \mathrm{Sym}(\mathbb{N})$ where $\mathrm{Fin}(\mathbb{N})$ is the group of permutations that fix all but finitely many elements of $\mathbb{N}$, we have a corresponding homeomorphism $\boldsymbol{\sigma} : \widehat{\mathbb{R}}^{\mathbb{N}} \to \widehat{\mathbb{R}}^{\mathbb{N}}$ given by

$$\boldsymbol{\sigma}(\mathbf{x}) = \left(x_{\sigma(n)}\right)_{n\in\mathbb{N}} \text{ for all } \mathbf{x} \in \widehat{\mathbb{R}}^{\mathbb{N}}$$

(c) Show that $A_q \Delta \boldsymbol{\sigma}[U]$ is meager in $\widehat{\mathbb{R}}^{\mathbb{N}}$ for all $\sigma \in \mathrm{Fin}(\mathbb{N})$. Using this fact, deduce that

$$A_q \Delta \bigcup_{\sigma\in\mathrm{Fin}(\mathbb{N})} \boldsymbol{\sigma}[U]$$

is meager in $\widehat{\mathbb{R}}^{\mathbb{N}}$.

Now, if $U = \emptyset$, then A_q is meager in $\widehat{\mathbb{R}}^{\mathbb{N}}$.

(d) Show that if $U \neq \emptyset$, then $\bigcup_{\sigma\in\mathrm{Fin}(\mathbb{N})} \boldsymbol{\sigma}[U]$ is dense in $\widehat{\mathbb{R}}^{\mathbb{N}}$ and as a result of this, A_q^c is meager in $\widehat{\mathbb{R}}^{\mathbb{N}}$.

This concludes the proof of the lemma. ■

Returning the proof of the main theorem, let $S = \{q \in \mathbb{Q} : \ A_q \text{ is meager in } \widehat{\mathbb{R}}^{\mathbb{N}}\}$.

(e) Show that $S \neq \emptyset$ and $S \neq \mathbb{Q}$.

Having shown this, set $z = \sup(S)$.

(f) Show that $B = \{\mathbf{x} \in \mathbb{R}^{\mathbb{N}} : \varphi(\mathbf{x}) = z\}$ is comeager in $\widehat{\mathbb{R}}^{\mathbb{N}}$.

It now follows from the Baire category theorem that B must be dense in $\widehat{\mathbb{R}}^{\mathbb{N}}$.

(g) Show that there exists $\mathbf{a} \in B$ such that $\varphi(\mathbf{a}) \in \{a_n\}_{n\in\mathbb{N}}$. ■

This finishes the final exam and our story of why we cannot ever come up with a diagonalization procedure that is both Borel and uniform.