MATH 779, Set Theory, Homework V and VI: Condensating on a minimal universe and how to get diamonds without force

1. (5+10 pts) Let L_α denote the α^{th} level of Gödel's constructible universe for each ordinal α.
a) Show that $|L_\alpha| = |\alpha|$ for all α.
b) Let $\mathcal{M} \preccurlyeq (L_{\omega_1}, \in)$. Show that M is transitive. (**Hint.** Show that any element of M is countable and that, in M, there is a definable surjection from ω to this subset.)

2. (15 pts) Recall that Gödel's condensation lemma states the following: For every limit ordinal δ, if $\mathcal{M} \preccurlyeq (L_\delta, \in)$, then the transitive collapse of M is equal to L_γ for some $\gamma \leq \delta$. Prove Gödel's condensation lemma.[1]

3. (10+10+10 pts) Let SM denote the statement that there exists a transitive model of ZF.
a) In ZF+SM, show that there exists $\delta < \omega_1$ such that L_δ is the minimal transitive model of ZFC, that is, $L_\delta \subseteq M$ for every transitive model M of ZFC.
b) Show that $L_\delta \models \text{ZFC} + \text{V=L} + \neg\text{SM}$.
c) Prove the relative consistency statement $\text{Con(ZFC)} \rightarrow \text{Con(ZFC+}\neg\text{SM)}$.

4. (60 pts)
a) Read the proof of $V = L$ implies $\diamondsuit$ from Jech's book.[2]
b) Prove that $V = L$ implies $\diamondsuit_\kappa$ for all uncountable regular κ.

[1] Clearly, I am not expecting you to prove the condensation lemma on your own. Rather, the purpose of this question is for you to search relevant sources, read a proof and rewrite the proof in your own phrasing.

[2] You need not hand in a solution for this part of the problem