MATH 779, Set Theory, Homework IV: Ramsey's ultrafilters and how to get diamonds

1. (10+10+15(+15 bonus) pts) Let Fin be the equivalence relation on $\mathcal{P}(\omega)$ defined by

$$A \text{ Fin } B \text{ if and only if } |A \Delta B| < \aleph_0$$

Let $\mathcal{P}_{\text{inf}}(\omega)$ denote the set of infinite subsets of ω. Consider the forcing poset

$$(\mathbb{P}, \leq, \mathbf{1}) = (\mathcal{P}_{\text{inf}}(\omega)/\text{Fin}, \subseteq^*, [\omega])$$

where $\subseteq^*$ is given by $[A] \subseteq^* [B]$ if and only if $|A \setminus B| < \aleph_0$.

Let $M \models ZFC$ be a countable transitive model. Let $G \subseteq \mathbb{P}^M$ be a $(\mathbb{P}, \leq, \mathbf{1})^M$-generic filter over M. Set $\mathcal{U} = \bigcup G$.

a) Show that $M[G] \models \mathcal{U}$ is a filter on ω.
b) Show that the forcing notion $(\mathbb{P}, \leq, \mathbf{1})$ is ω_1-closed.
c) Show that $M[G] \models \mathcal{U}$ is an ultrafilter on ω.
d) Show that $M[G] \models \mathcal{U}$ is a Ramsey ultrafilter ω.[1]

2. (5+5+15 pts) Consider the forcing poset $(\mathbb{P}_\diamondsuit, \leq, \emptyset)$ where

$$\mathbb{P}_\diamondsuit = \{\langle A_\alpha : \alpha < \beta \rangle \ \mid \ \beta < \omega_1 \text{ and } A_\alpha \subseteq \alpha \text{ for all } \alpha < \beta\}$$

and

$$\langle A_\alpha : \alpha < \beta \rangle \leq \langle B_\alpha : \alpha < \gamma \rangle \text{ iff } \beta \geq \gamma \text{ and } A_\alpha = B_\alpha \text{ for every } \alpha < \gamma$$

Let $M \models ZFC$ be a countable transitive model. Let $G \subseteq \mathbb{P}_\diamondsuit^M$ be a $(\mathbb{P}_\diamondsuit, \leq, \emptyset)^M$-generic filter over M.

a) Show that the forcing notion $(\mathbb{P}_\diamondsuit, \leq, \emptyset)$ is ω_1-closed.
b) Show that $\omega_1^{M[G]} = \omega_1^M$.
c) Show that $M[G] \models \diamondsuit$.

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[1] Recall that an ultrafilter $\mathcal{U}$ on ω is called a *Ramsey ultrafilter* if and only if for every partition $\{A_n : n \in \omega\}$ of ω such that $A_n \notin \mathcal{U}$, there exists $A \in \mathcal{U}$ such that $|A \cap A_n| = 1$ for every $n \in \omega$. This condition is also equivalent to the following: For every $f : [\omega]^2 \to \{0, 1\}$ there exists $B \in \mathcal{U}$ such that f is constant on $[B]^2$, that is, the ultrafilter $\mathcal{U}$ provides witnesses for infinite Ramsey theorem for two colors.