MATH 779, Set Theory, Homework III: Fragments of ZFC, inaccessibles and reflection

1. (10+10+10 pts) Prove that

- $V_\omega \models$ ZFC-Infinity
- $V_{\omega+\omega} \models$ ZFC-Replacement
- $H_{\omega_1} \models$ ZFC-Power Set[1]

2. (15 pts) Prove that if there exists an inaccessible cardinal, then there exists a transitive model M such that

$$M \models \text{ZFC} + \text{"There exists no inaccessible cardinals"}$$

Using the appropriate arguments in your proof, conclude that, in ZFC, we can prove the relative consistency statement

$$\text{Con(ZFC)} \rightarrow \text{Con(ZFC+"There exists no inaccessible cardinals")}$$

but the relative consistency statement

$$\text{Con(ZFC)} \rightarrow \text{Con(ZFC+"There exists an inaccessible cardinal")}$$

cannot be proven.

3. (15 pts) Prove the following variation of the reflection principle for the constructible hierarchy[2]: Let $\varphi(x_1, \ldots, x_n)$ be a formula in the language of set theory. Show that there exists arbitrarily large limit ordinals α such that we have

$$\varphi^L(x_1, \ldots, x_n) \longleftrightarrow \varphi^{L_\alpha}(x_1, \ldots, x_n)$$

[1] For each set x, let $trcl(x)$ denote the transitive closure of x and define the class of sets of hereditarily countable sets as $H_{\omega_1} = \{x \ : \ |trcl(x)| < \omega_1\}$.

[2] First, learn the proof of the standard form of the reflection principle from Jech's book.