MATH 779, Set Theory, Homework II: Clubs and Jensen's diamonds

1. (15 pts) Let ♦ be the following statement: There exists a sequence $(A_\alpha)_{\alpha<\omega_1}$ such that $A_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$ and

$$\text{for all stationary } A \subseteq \omega_1 \text{ there exists } \alpha \in A \text{ with } A_\alpha = A \cap \alpha$$

Show that ZFC+♦ is inconsistent, that is, ♦ is false.

2. (15 pts) Let κ be a regular uncountable cardinal and $\diamondsuit_\kappa$ be the following statement: There exists a sequence $(A_\alpha)_{\alpha<\kappa}$ such that $A_\alpha \subseteq \alpha$ for all $\alpha < \kappa$ and

$$\{\alpha < \kappa : \ A \cap \alpha = A_\alpha\} \text{ is stationary in } \kappa \text{ for all } A \subseteq \kappa$$

Show that $\diamondsuit_\kappa$ implies $2^{<\kappa} = \kappa$.

3. (15 pts) Alice and Bob are playing a two-player game of length ω by alternately choosing a decreasing sequence of stationary subset of ω_1. In other words, a typical play of the game looks as follows.

Alice	A_0		A_2		A_4		$\ldots$
Bob		A_1		A_3		A_5	$\ldots$

where $A_n \subseteq \omega_1$ is stationary and $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$. The winning condition is that Bob wins the game if and only if $\bigcap_{n\in\mathbb{N}} A_n$ has at most one element. Show that Bob has a winning strategy.

4. (15 pts) Let ♣ be the following statement, which is a weakening of $\diamondsuit$: There exists a sequence $(A_\alpha)_{\alpha<\kappa}$ such that

- A_α is cofinal in α, for every $\alpha < \omega_1$, and
- For every unbounded $A \subseteq \omega_1$, there exists $\delta < \omega_1$ such that $A_\delta \subseteq A$.

Show that $\diamondsuit$ holds if and only if ♣ and CH hold.