MATH 779, Set Theory, Homework I: A night at Martin's Opera

1. (10+10 pts) Consider the following classical theorems of Cantor, the former of which is usually proven by a back-and-forth argument and the latter of which is usually proven by diagonalization.

a) Any two countable dense linear orders without endpoints are isomorphic.
b) ${}^{\omega}2$ is not countable.

In this question, your task is to provide alternative proofs of these theorems by

- constructing an appropriate partial order relation $(\mathbb{P}, \leq)$, and
- applying $MA(\omega)$ to obtain a filter G, and then
- using G to construct a suitable object that proves the result.

Here, the idea is to use $MA(\omega)$, which is provable in ZFC, as a replacement of the aforementioned procedures.

2. (10+10 pts) For this question, fix a non-principal ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\omega)$ on ω. A subset $\mathcal{B} \subseteq \mathcal{U}$ is said to generate $\mathcal{U}$ if for every $U \in \mathcal{U}$ there exists $B \in \mathcal{B}$ such that $B \subseteq U$. Consider the cardinal number $d_{\mathcal{U}} = \min\{|\mathcal{B}| : \mathcal{B} \text{ generates } \mathcal{U}\}$.

a) Show that $\aleph_0 < d_{\mathcal{U}} \leq 2^{\aleph_0}$.
b) Show that, under Martin's axiom, we have $d_{\mathcal{U}} = 2^{\aleph_0}$.

3. (10+10 pts) An Aronszajn tree $(T, \preceq)$ is said to be *special* if there exists a map $f : T \to \mathbb{Q}$ such that $x \prec y$ implies $f(x) < f(y)$ for all $x, y \in T$.

a) Construct a special Aronszajn tree.
b) Show that, under Martin's axiom and the negation of Continuum Hypothesis, every Aronszajn tree is special.